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D-branes, matrix theory and K-homology

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Abstract: In this paper, we study a new matrix theory based on non-BPS D-instantons in type-IIA string theory and D-instanton–anti D-instanton system in type-IIB string theory, which we call K-matrix theory. The theory correctly incorporates the creation and annihilation processes of D-branes. The configurations of the theory are identified with spectral triples, which are the noncommutative generalization of riemannian geometry á la Connes, and they represent the geometry on the world-volume of higher dimensional D-branes. Remarkably, the configurations of D-branes in the K-matrix theory are naturally classified by a K-theoretical version of homology group, called K-homology. Furthermore, we argue that the K-homology correctly classifies the D-brane configurations from a geometrical point of view. We also construct the boundary states corresponding to the configurations of the K-matrix theory, and explicitly show that they represent the higher dimensional D-branes.

Keywords: D-branes, M(atrix) Theories, Non-Commutative Geometry.

Contents

1. Introduction

The recent development of non-BPS systems shows that it is essential to incorporate the creation and annihilation process of non-BPS D-branes or D-brane–anti D-brane pairs in string theory. For example, as shown in [1], there are some examples that D-branes wrapped on non-trivial cycles decay through the creation and annihilation process. In fact, taking these process into account, D-brane charges are successfully classified by K-theory [2], which shows that (co)homology groups are no longer basic tools for the classification of topologically stable configurations of D-branes. It also suggests that the usual description of RR-fields as differential forms is, in general, insufficient to correctly describe the background of string theory [3]. Therefore, it seems to be very important to find a non-perturbative formulation of string theory, in which the creation and annihilation process of D-branes is correctly incorporated, as in the second quantization of field theory.

The K-theory structure of the theory appears most naturally in the world-volume gauge theory of space-time filling brane systems, i.e. non-BPS D9-brane system in type-IIA string theory [4] and D9-$\overline{\text{D9}}$ system in type-IIB string theory [2]. However, since the ten dimensional gauge theories are non-renormalizable, it is hard to consider this kind of theory as a fundamental theory. So, lower dimensional systems are preferable for our purpose. One of the most interesting possibilities is the lowest dimensionalq case, namely matrix theory. The matrix formulations of type-II string theory or M-theory are proposed in [5, 6, 7] etc. However, the K-theory structure is not clear in the framework of these matrix theories. One of the reasons is that the matrix model of [5], for example, is formulated as the theory of infinite number of D-particles, which corresponds to the infinite momentum frame in M-theory and there are no anti D-particles. Similarly, K-theory structure of the IIB matrix theory [6] should be realized in a completely different way from the D9-$\overline{\text{D9}}$ system since it is not based on the D-$\overline{\text{D}}$ system.

In this paper, we propose a new type of matrix theory based on non-BPS D-instantons in type-IIA theory and D-instanton–anti D-instanton system in type-IIB theory. If one prefer Minkowski space-time rather than euclidean one, it could be more interesting to consider matrix theories based on D-particle–anti D-particle system in type-IIA theory and non-BPS D-particles in type-IIB theory. But, since the formulation of the latter is quite analogous to the former, we mainly deal with the former for simplicity.

It is tempting to consider such matrix theory as the fundamental theory of covariant type-II string theory. Unfortunately, however, we do not know the precise form of the action of the theory. We leave the attempt to give a precise formulation and the argument about the consistency of the theory as future problems. Nevertheless, as we will see in the following sections, it is possible to analyze the topologically stable configurations of the theory. The configurations of the theory can be interpreted as the higher dimensional D-branes, and we will show in section 4 that they are classified by a K-homology group, which is a dual of K-theory group. Therefore, the theory successfully recovers the K-theory structure. For this reason, we would like to refer to these matrix theories as K-matrix theory.

At first sight, it seems to be strange that the K-homology naturally appears in the K-matrix theory instead of the K-theory. However, it turns out that the configurations of the K-matrix theory represent the world-volume of D-branes, which should be classified by some kinds of homology theory. The K-homology carries not only the information of cycles but also that of gauge bundles on them. Hence the K-homology is a natural candidate for the classification of D-brane configurations.

In the formulation of matrix theories, geometry does not play an important role, since the world-volume manifold is just a set of zero dimensional points. There are no non-trivial topology, metric, gauge bundle, and so on. Instead, the algebra among the matrix variables becomes important. In the K-matrix theory, the size of the matrix variables are infinity from the beginning, so that arbitrary numbers of non-BPS D-instantons or D-instanton–anti D-instanton pairs can be created. Therefore, the matrices should be thought of as linear operators acting on an infinite dimensional Hilbert space. One of the interesting features in the K-matrix theory is that the geometric information is hidden in this operator algebra. In fact, it is well-known that there is a one-to-one correspondence between operator algebras and topological spaces. More precisely, the $*$-isomorphism classes of commutative C^*-algebras are in one-to-one correspondence with the homeomorphism classes of locally compact Hausdorff spaces. Here, a C^*-algebra is a norm closed self-adjoint subalgebra of the bounded operator algebra $\mathbf{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. This fact is the starting point of the noncommutative geometry. Remarkably, the variables in the K-matrix theory have a direct interpretation in terms of the noncommutative geometry. We will argue in section 3 that these variables define a spectral triple, which is introduced by Connes to define a noncommutative generalization of the riemannian geometry [8], and explain that the spectral triples represent the configurations of D-branes, that is, the geometry represented by a spectral triple is nothing but the world-volume geometry of the corresponding D-branes.

The K-homology is defined as equivalence classes of the spectral triples under some equivalence relations. The equivalence relations also have physical interpretations, that is, continuous deformation (homotopy equivalence), gauge equivalence and creation and annihilation of non-BPS D-instantons or D-instanton–anti D-instanton pairs. These observations suggest that the D-brane configurations in the K-matrix theory are classified by the K-homology groups. We also have a topological description of K-homology, which enables us to interpret the classification in terms of geometry of the D-brane world-volume. We will explain these facts in detail in section 4.

We also analyze these facts using boundary state approach of D-branes along the line with the work of [9], in which boundary states of higher dimensional D-branes are constructed from the boundary states of noncommutative configurations of D-instantons in bosonic string theory. We can formally construct a boundary state corresponding to each configuration of the K-matrix theory. One of the merits of this approach is that we can avoid the ambiguity in the action of the K-matrix theory. Actually, this approach is closely related to the BSFT approach and we can implicitly use the BSFT action by analyzing the boundary states. Thanks to this fact, as we will see in section 5, we can explicitly show that a canonical choice of the spectral triples represents a higher dimensional D-brane. In other words, we can explicitly see how the operators in the K-matrix theory represent the geometry of D-branes using string theory. This gives another viewpoint for our proposal that D-branes are represented by spectral triples, and also for the correspondence between operator algebras and topological spaces.

The paper is organized as follows. In section 2, we outline the basic structure of the K-matrix theory, and explain how the configurations with finite action can be expressed. Section 3 deals with the geometric interpretation of the configurations. We explain that they are related to the spectral triples and interpreted as the configurations of higher dimensional D-branes. In section 4, we claim that the D-brane configurations in the K-matrix theory are classified by K-homology. Some comments on the generalization to the higher dimensional cases are made in section 4.5 from the viewpoint of KK-theory. We also discuss that the Chern-Simons terms are given by the Chern character of the K-homology. Section 5 is devoted to the boundary state approach. We will construct the boundary states of higher dimensional D-branes from the boundary state of non-BPS D-instantons with tachyon condensation. Note also that the calculation performed in this section provides an efficient way to obtain the correct tensions and RR-charges of D-branes made via tachyon condensation. Finally, we make some speculative discussions in section 6.

2. K-matrix theory

2.1 Type-IIA K-matrix theory

In type-IIA string theory, the lowest dimensional D-brane is the non-BPS D-instanton. In order to obtain the exact action of the theory explicitly, we may need powerful symmetries such as supersymmetry, conformal symmetry, etc. The nonlinearly realized supersymmetry with 32 supercharges, which is based on the idea that the vacuum with a non-BPS D-brane belongs to the spontaneously broken phase of the supersymmetry [10, 11], may be strong enough to determine the action, as calculated in [12] up to some order. Up to now, however, we do not know how to write down the action of the theory exactly. Nevertheless, as we will see, we can extract some topological information of the theory, on which we mainly focus in this paper, without knowing the detailed structure of the theory.

Let us summarize the main ingredients of the non-BPS D-instanton theory. The gauge group of the theory with N non-BPS D-instantons is $\mathrm{U}(N)$. The bosonic part consists of ten scalar fields Φ^μ ($\mu = 0, \ldots, 9$) and a tachyon T. They are self-adjoint (hermitean) matrices and belong to the adjoint representation of the gauge group.[1]

The important point is that in order to incorporate creation and annihilation of the non-BPS D-instantons, we must take the limit $N = \infty$, so that arbitrary numbers of non-BPS D-instantons can be created. Therefore, the vector space, on which matrices Φ^μ and T are represented, is an infinite dimensional vector space. We assume that this vector space is a Hilbert space $\mathcal{H}$. This Hilbert space should be separable, i.e. it has countably many orthonormal basis, since there is a one-to-one correspondence between the basis of $\mathcal{H}$ and the Chan-Paton indices of non-BPS D-instanton. Note that since every infinite dimensional separable Hilbert space is isomorphic to $l^2(\mathbf{N})$, we can uniquely associate the Hilbert space $\mathcal{H}$ with the Chan-Paton indices of the non-BPS D-instantons up to isomorphism. Then Φ^μ and T are regarded as linear operators acting on the Hilbert space $\mathcal{H}$.

The action is basically obtained through the dimensional reduction of the non-BPS D9-brane action. The kinetic terms and the tachyon potential are roughly given as

$$S(T, \Phi^\mu) \sim \mathrm{Tr}\left(e^{-T^2}[\Phi^\mu, \Phi^\nu]^2 + e^{-T^2}[\Phi^\mu, T]^2 + e^{-T^2} + \cdots\right). \qquad (2.1)$$

[1] Throughout this paper, we only consider the tachyons and massless modes. Massive modes are considered to be integrated out or neglected.

Actually, using the boundary string field theory [13]–[16], the action for non-BPS D-instantons [12, 17, 13] is calculated as

$$S = -\sqrt{2}T_{-1}\,\mathrm{Tr}\left(e^{-\frac{1}{4}T^2}\sqrt{\det(\delta_{\mu\nu} - i[\Phi_\mu,\Phi_\nu])}\,\mathcal{F}\left[\frac{1}{4\pi}\mathcal{G}^{\mu\nu}[\Phi_\mu,T][\Phi_\nu,T]\right]\right), \tag{2.2}$$

where $\mathcal{G}^{\mu\nu} \equiv (1/1 - i[\Phi_\mu,\Phi_\nu])^{(\mu\nu)}$ and $\mathcal{F}[x] \equiv 4^x x(\Gamma(x))^2/2\Gamma(2x) = 1 + 2(\log 2)x + O(x^2)$. Here, $(\mu\nu)$ indicates the symmetrization. Though the action is not exact, i.e. the higher commutator terms are neglected, the action (2.2) may be a rather good starting point to consider the non-BPS D-branes. In fact, the action successfully describes the tachyon condensation and contains the Dirac-Born-Inferd type action.

From this it is reasonable to expect that the full action has the form (2.1) and then the action is roughly estimated by the following inequality.

$$|S(T,\Phi^\mu)| \leq \mathrm{Tr}\, e^{-T^2}\left(\parallel [\Phi^\mu,\Phi^\nu] \parallel^2 + \parallel [\Phi^\mu,T] \parallel^2 + 1\right) + \cdots, \tag{2.3}$$

where $\parallel\cdot\parallel$ is the operator norm. In order to obtain configurations with finite action, it seems to be natural to require

$$\mathrm{Tr}\; e^{-T^2} < \infty\,, \qquad \parallel [\Phi^\mu,\Phi^\nu] \parallel < \infty\,, \qquad \parallel [\Phi^\mu,T] \parallel < \infty\,, \tag{2.4}$$

that is, $[\Phi^\mu,\Phi^\nu]$ and $[\Phi^\mu,T]$ are bounded operators for $\mu,\nu = 0,1,\ldots,9$, and e^{-T^2} is traceclass.

In particular, the first condition in (2.4) implies that the tachyon T is not a bounded operator. To see this fact, let $\{\lambda_n\}$ be the set of eigenvalues of the operator T. In order for the potential Tr e^{-T^2} to be finite, each eigenvalue should have finite multiplicity, and $|\lambda_n| \to \infty$ as $n \to \infty$. Since the norm of an operator is larger than or equal to its largest eigenvalue, the norm of T diverges to infinity.

To avoid this infinity, it may be convenient to use the bounded operator

$$T_b = \frac{T}{\sqrt{1+T^2}} \tag{2.5}$$

normalized such that $T_b^2 = 1$ is the minimum of the potential. In this normalization, the eigenvalues of T_b^2 accumulates to 1 in order to obtain a finite energy configuration. In other words, we require $T_b^2 - 1$ to be a compact operator. Here an operator K on $\mathcal{H}$ is said to be compact if it has an expansion

$$K = \sum_{n\geq 0}\mu_n\,|\psi_n\rangle\,\langle\phi_n|\;, \tag{2.6}$$

where, $\mu_n \to 0$ as $n \to \infty$, $\{\psi_n\}_{n\in\mathbf{N}}$ and $\{\phi_n\}_{n\in\mathbf{N}}$ are orthonormal sets.

Similarly, the requirement that $[\Phi^\mu,T]$ $(\mu = 0,1,\ldots,9)$ are bounded operators implies that $[\Phi^\mu,T_b]$ are compact operators. This condition is analogous to the condition $D_\mu T(x) \to 0$ at infinity in usual field theory. Therefore we should, at least, require the self-adjoint bounded operator T_b to satisfy the conditions

$$T_b^2 - 1 \in \mathbf{K}(\mathcal{H})\,, \qquad [\Phi^\mu,T_b] \in \mathbf{K}(\mathcal{H})\,, \qquad (\mu = 0,1,\ldots,9)\,, \tag{2.7}$$

where $\mathbf{K}(\mathcal{H})$ denotes the set of compact operators on $\mathcal{H}$.

Strictly speaking, we don't know whether these conditions for the operators Φ^μ and T are necessary nor sufficient ones for the finiteness of the action, since we don't know the exact action of the theory. In the following sections, we will see that our proposal is highly plausible. They beautifully fit the mathematical framework of noncommutative geometry and K-homology. We will also examine some examples and see how they work.

2.2 Type-IIB K-matrix theory

The argument in the previous subsection can also be applied to type-IIB string theory. There are BPS D-instantons in type-IIB string theory, and the matrix theory related to the D-instantons is constructed in [6]. In order to incorporate creation and annihilation of D-instanton–anti D-instanton pairs in the theory, we should add degrees of freedom of the anti D-instantons. Thus the matrix theory we consider here is based on the D-instanton–anti D-instanton system. The world-point theory of N D-instantons and M anti D-instantons has $\mathrm{U}(N) \times \mathrm{U}(M)$ gauge symmetry. There are ten pairs of scalar fields Φ^μ, $\overline{\Phi}^\mu$ ($\mu = 0, \ldots, 9$) and a tachyon field T. Φ^μ, $\overline{\Phi}^\mu$ and T are in (adjoint,1), (1,adjoint) and (N, M) representation of the gauge group $\mathrm{U}(N) \times \mathrm{U}(M)$, respectively.

We take both N and M to be infinity. The Chan-Paton Hilbert space should be $\mathbb{Z}_2$-graded as $\widehat{\mathcal{H}} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)}$, where the basis of $\mathcal{H}^{(0)}$ and $\mathcal{H}^{(1)}$ is in one-to-one correspondence with the D-instantons and the anti D-instantons, respectively. The scalar fields Φ^μ and $\overline{\Phi}^\mu$ are operators acting on $\mathcal{H}^{(0)}$ and $\mathcal{H}^{(1)}$, respectively, while the tachyon T is an operator from $\mathcal{H}^{(0)}$ to $\mathcal{H}^{(1)}$.

As discussed in the previous subsection, there are some constraints on these operators, which are analogous to (2.4) and (2.7) for the IIA K-matrix theory. If we use the normalized tachyon field T_b, such that the minimum of the potential is given by $T_b^* T_b = T_b T_b^* = 1$, the same argument as in (2.7) implies that T_b should be an element of $\mathbf{B}(\mathcal{H}^{(0)}, \mathcal{H}^{(1)})$, bounded linear operators from $\mathcal{H}^{(0)}$ to $\mathcal{H}^{(1)}$, which satisfies

$$T_b^* T_b - 1 \in \mathbf{K}(\mathcal{H}^{(0)}) \,, \qquad T_b T_b^* - 1 \in \mathbf{K}(\mathcal{H}^{(1)}) \,, \tag{2.8}$$

$$T_b \Phi^\mu - \overline{\Phi}^\mu T_b \in \mathbf{K}(\mathcal{H}^{(0)}, \mathcal{H}^{(1)}) \,, \qquad (\mu = 0, 1, \ldots, 9) \,. \tag{2.9}$$

Here $\mathbf{K}(\mathcal{H}^{(0)}, \mathcal{H}^{(1)})$ denotes the set of compact operators from $\mathcal{H}^{(0)}$ to $\mathcal{H}^{(1)}$. The conditions (2.8) and (2.9) are again expected from the finiteness of the potential and kinetic terms of the tachyon, respectively.

We can rewrite these conditions just like (2.7) as

$$\widehat{F}^2 - 1 \in \mathbf{K}(\widehat{\mathcal{H}}) \,, \qquad [\widehat{\Phi}^\mu, \widehat{F}] \in \mathbf{K}(\widehat{\mathcal{H}}) \,, \qquad (\mu = 0, 1, \ldots, 9) \,, \tag{2.10}$$

where $\widehat{\Phi}^\mu = \mathrm{diag}(\Phi^\mu, \overline{\Phi}^\mu)$ and $\widehat{F} = \begin{pmatrix} & T_b^* \\ T_b & \end{pmatrix}$.

Note that there are certain similarities between this IIB K-matrix theory and the model of [18] which is based on the gauge theory on two points. Actually, the D-$\overline{\text{D}}$ pair in the K-matrix theory represents the discrete two points and the tachyon T corresponds to the Higgs field in [18, 8], though the precise action is not exactly the same.

2.3 Chern-Simons terms and D-brane configurations

Although we don't have the exact action for the K-matrix theory, we can use some exact results from the calculation in BSFT [13, 15, 16]. In particular, Chern-Simons terms are known exactly, at least in the case that the RR-fields are constant. (See [19, 15] for the comments on the corrections.)

Let us briefly review the Chern-Simons terms in the action of N non-BPS D-instantons in type-IIA string theory. First we introduce fermions ψ_1^μ, ψ_2^μ ($\mu = 0, \ldots, 9$), which represent SO(10, 10) gamma matrices, satisfying the anti-commutation relations

$$\{\psi_1^\mu, \psi_2^\nu\} = \delta^{\mu\nu}, \qquad \{\psi_1^\mu, \psi_1^\nu\} = \{\psi_2^\mu, \psi_2^\nu\} = 0. \tag{2.11}$$

Then the Chern-Simons term obtained in [16] can be written as

$$\begin{aligned} S_{\rm CS} &= \mathrm{Sym}\,\mathrm{Tr}_N\,\mathrm{Tr}_2\left(\sigma^1\,\mathrm{Tr}_\psi\left(\widehat{C}e^{Z^2}\right)\right) & (2.12)\\ &= \mathrm{Sym}\,\mathrm{Tr}_N\,\mathrm{Tr}_2\left(\sigma^1\,\mathrm{Tr}_\psi\left(\widehat{C}e^{\left(-T^2+\frac{1}{2}[\Phi^\mu,\Phi^\nu]\psi_2^\mu\psi_2^\nu+i[\Phi^\mu,T]\psi_2^\mu\sigma^1\right)}\right)\right), & (2.13)\end{aligned}$$

where

$$\begin{aligned} iZ &= -i\Phi^\mu\psi_2^\mu + T\sigma^1, & (2.14)\\ \widehat{C} &= \sum_n C_{\mu_1\cdots\mu_n}(\Phi)\,\psi_1^{\mu_1}\cdots\psi_1^{\mu_n}. & (2.15)\end{aligned}$$

Note that $\sigma^1 = \begin{pmatrix} & 1\\ 1 & \end{pmatrix}$ also behaves as a fermion.

Here Z is regarded as a 2×2 matrix, whose components are also matrices, and Tr_2 denotes the trace of the 2×2 matrices. Tr_N is the trace over the U(N) gauge indices and Tr_ψ stands for the trace over the SO(10, 10) gamma matrices. The symbol Sym in (2.13) means that we expand $\widehat{C}$ in the power series of Φ^μ and then symmetrize them with T^2, $[T, \Phi^\mu]$, $[\Phi^\mu, \Phi^\nu]$. We can show that the CS-term (2.13) is invariant under the gauge transformation of the RR-fields $C \to C + d\Lambda$, generalizing the proof in [20] for the CS-term of BPS D-branes including Myers' terms [21]. See appendix A for the detail.

Taking the formal limit $N \to \infty$, we obtain the Chern-Simons term for the IIA K-matrix theory. The trace Tr_N is replaced by the trace $\mathrm{Tr}_{\mathcal{H}}$ over the Hilbert space $\mathcal{H}$.

Let us consider a simple situation such that $C_{\mu_1\ldots\mu_n} = $ const. (n: odd) is the only non-zero RR-field, and $[\Phi^\mu, \Phi^\nu] = 0$ for $^\forall\mu,\nu$. Then, using the formula

$$e^{A+B} = e^A + \int_0^1 dt\, e^{(1-t)A}Be^{tA} + \int_0^1 dt_1 \int_0^{t_1} dt_2\, e^{(1-t_1)A}Be^{(t_1-t_2)A}Be^{t_2A} + \cdots, \tag{2.16}$$

we can rewrite (2.13) as

$$\begin{aligned} S_{\rm CS} = C_{\mu_1\ldots\mu_n} &\int_{0\le t_n\le\cdots\le t_1\le 1} dt_1\cdots dt_n\,\times\\ &\times \mathrm{Tr}_{\mathcal{H}}\left(e^{-(1-t_1)T^2}[T,\Phi^{\mu_1}]e^{-(t_1-t_2)T^2}\cdots[T,\Phi^{\mu_n}]e^{-t_nT^2}\right). \end{aligned} \tag{2.17}$$

Hölder's inequality on the integrand in (2.17) implies that

$$|S_{CS}| \leq \frac{1}{n!}|C_{\mu_1\dots\mu_n}| \operatorname{Tr}_{\mathcal{H}}\left(e^{-T^2}\right) \prod_{k=1}^{n} \| [T, \Phi^{\mu_k}] \| \ . \tag{2.18}$$

Thus the CS-term is finite for the operators satisfying (2.4). In general, the CS-term of a D-brane is estimated as $|S_{CS}| \sim |C_{\mu_1\dots\mu_n}|V$, where V is the volume of the world-volume. Actually, as we will see soon, $\operatorname{Tr}_{\mathcal{H}}(e^{-T^2})$ is proportional to the volume, and the first condition in (2.4), which is called the θ-summability condition, is related to the compactness of the world-volume. Thus, it could be relaxed if we allow the infinite volume configurations.

Let us explain these facts in an explicit example, that is the D-brane solution given in [22]. We consider the Chan-Paton Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{2m+1}) \otimes S$, where S is 2^m dimensional vector space of SO$(2m+1)$ spinors. The D$(2m)$-brane configuration is given by

$$T = u\, D = u \sum_{\alpha=0}^{2m} \widehat{p}_\alpha \otimes \gamma^\alpha\,, \tag{2.19}$$

$$\Phi^\alpha = \widehat{x}^\alpha \otimes 1 \quad (\alpha = 0, \dots, 2m)\,, \qquad \Phi^i = 0 \quad (i = 2m+1, \dots, 9)\,, \tag{2.20}$$

where $\widehat{x}^\alpha$ is defined by multiplication of x^α and $\widehat{p}_\alpha = -i\partial/\partial x^\alpha$ is a differential operator, both acting on $L^2(\mathbb{R}^{2m+1})$. Note that the tachyon field T is a Dirac operator D up to normalization. It can be shown that this becomes an exact BPS D$(2m)$-brane solution if we take $u \to \infty$, although this configuration with finite u still represents a D$(2m)$-brane. Inserting this configuration into the Chern-Simons term (2.17) with $n = 2m+1$, we obtain

$$S_{\rm CS} = u^{2m+1} C_{01\cdots 2m} \operatorname{Tr}_{\mathcal{H}}\left(e^{-u^2 D^2}\right). \tag{2.21}$$

We can evaluated $\operatorname{Tr}_{\mathcal{H}}(e^{-u^2D^2})$ as

$$\operatorname{Tr}_{\mathcal{H}}\left(e^{-u^2D^2}\right) = 2^m \int d^{2m+1}k \, \langle k| e^{-u^2k^2} |k\rangle \tag{2.22}$$

$$= \frac{\mu_{2m}}{u^{2m+1}} \int d^{2m+1}x\,, \tag{2.23}$$

where $\mu_{2m} = 1/(2^{m+1}\sqrt{\pi}^{2m+1})$ is a numerical constant.

Therefore the trace of the operator $e^{-u^2D^2}$ is proportional to the volume factor $\int dx$ and diverges for the infinite volume case. In this paper, we will tacitly compactify the space-time and consider θ-summable family of tachyon operators in order to avoid this divergence.

It is worthwhile to mention that the $S_{\rm CS}$ does not depend on the parameter u as expected, and the coefficient μ_{2m} gives the correct value of the coupling between the D$(2m)$-brane and the RR $(2m+1)$-form field [22].

3. Spectral triples and D-branes

In section 2, we explained the possible configurations of the K-matrix theory in purely analytic language. In this section, we claim that these configurations correspond to the configurations of various D-branes embedded in the space-time, and explain how to extract geometric information out of the operators Φ^μ and T introduced in section 2.1. In particular, as we will see in this section, each configuration in the K-matrix theory defines a spectral triple, which is the analytic analog of the riemannian manifold, and we can use the techniques developed in noncommutative geometry.

3.1 Topology and algebra of D-branes

Let $\widehat{\mathcal{A}}$ be the algebra generated by the operators Φ^μ for a configuration in the IIA K-matrix theory. Note that $\widehat{\mathcal{A}}$ is an involutive algebra of operators acting on a Hilbert space $\mathcal{H}$, since it is equipped with a star-operation (hermitean conjugation). When $\widehat{\mathcal{A}}$ is a subalgebra of the bounded operator algebra $\mathbf{B}(\mathcal{H})$ for the Hilbert space $\mathcal{H}$, $\widehat{\mathcal{A}}$ can be thought of as a C^*-algebra by taking the completion.[2]

Let us first consider the case that $\widehat{\mathcal{A}}$ is a commutative C^*-algebra. The Gel'fand-Naimark theorem states that every *commutative* C^*-algebra is of the form $C_0(M)$, i.e. the space of continuous complex functions on some locally compact Hausdorff space M, vanishing at infinity. (If M is compact, $C_0(M)$ is equal to $C(M)$ which is the space of continuous complex functions on M.) Note that the norm of an element of $C(M)$ is defined by its supremum value, and hence there are unbounded elements in $C(M)$ if M is not compact.

Moreover, it is known that the category of commutative C^*-algebras is in one-to-one correspondence with the category of topological spaces (locally compact Hausdorff spaces). A point $x \in M$ on the space M corresponds to a character of $\widehat{\mathcal{A}}$, which is a $*$-homomorphism $\phi_x : \widehat{\mathcal{A}} = C_0(M) \to \mathbb{C}$. Suppose that $\widehat{\mathcal{A}}$ is generated by mutually commuting operators Φ^μ ($\mu = 0, 1, \dots, 9$), the character ϕ_x of $\widehat{\mathcal{A}}$ is given by picking up one of the elements x from the joint spectrum of $(\Phi^0, \Phi^1, \dots, \Phi^9)$. This agrees with the standard interpretation that the eigenvalues of the matrix Φ^μ represents the position of the non-BPS D-instantons along the space-time coordinate x^μ. When the spectrum of $(\Phi^0, \Phi^1, \dots, \Phi^9)$ agrees with the manifold M embedded in $\mathbb{R}^{10}$, we can say that M is paved with non-BPS D-instantons. Therefore, the topological space M is interpreted as the world-volume of higher dimensional objects made from infinite number of non-BPS D-instantons. Actually, as it will become clear in the next section, M is identified with the world-volume of higher dimensional D-branes.

One major problem for this interpretation is that Φ^μ are not necessarily bounded operators and the algebra generated by Φ^μ may not be a C^*-algebra. For example, if we take $\mathcal{H} = L^2(\mathbb{R}^n)$ and $\Phi^\mu = \widehat{x}^\mu$ ($\mu = 1, \dots, n$), which is the multiplication operator $\Phi^\mu : f(x) \in \mathcal{H} \to x^\mu f(x)$, the spectrum of Φ^μ is not bounded and hence Φ^μ is an unbounded operator. This happens when we consider the D-branes with non-compact world-volume. One way to avoid this problem is to compactify the manifold $\mathbb{R}^n$ to S^n. We can achieve this by replacing $\mathcal{H} = L^2(\mathbb{R}^n)$ with $\mathcal{H} = L^2(S^n)$ and let $\widehat{\mathcal{A}}$ be the algebra generated by $\Phi^\mu = \widehat{x}^\mu$ ($\mu = 0, \dots, n$) with a relation $\sum_{\mu=0}^n (\Phi^\mu)^2 = R^2$, that is $\widehat{\mathcal{A}} = C(S^n)$. Another way

[2]A C^*-algebra is a norm closed self-adjoint subalgebra of the bounded operator algebra $\mathbf{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

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is to restrict ourselves to the subalgebra whose elements are of the form $f(\Phi^{\mu})$ with some cut off function $f \in C_0(\mathbb{R}^n)$, and set $\widehat{\mathcal{A}} = C_0(\mathbb{R}^n)$. With these modification, we assume that $\widehat{\mathcal{A}}$ consists of bounded operators and makes a C^*-algebra.

Another problem, which is a common issue in matrix theories, is that the interpretation that the spectrum of Φ^{μ} represent the coordinate of x^{μ} axis seems to be possible only when the manifold has a global coordinate system $(x^0, x^1, \ldots, x^9)$, (or its quotient such as torus, orbifolds and so on). It is not clear how to describe the theory when the background manifold is topologically non-trivial. An ad hoc resolution for this problem is obtained by formally embedding the manifold to a higher dimensional euclidean space $\mathbb{R}^N$ $(N \geq 10)$ and introducing Φ^{μ} $(\mu = 0, \ldots, N-1)$ as the scalar field corresponding to the fluctuation of x^{μ} direction, which are subjected to some constraints representing the embedded manifold. More sophisticated description of the K-matrix theory in general background will be discussed elsewhere [23]. (See also [24, 25, 26].)

3.2 Spectral triples

In the last subsection, we saw that the C^*-algebra $\widehat{\mathcal{A}}$ corresponds to a topological space M, which is interpreted as the world-volume of D-branes. Then, what is the geometric interpretation of the tachyon operator T? In this subsection, we claim that the triple $(\mathcal{H}, \widehat{\mathcal{A}}, T)$ can be interpreted as the spectral triple, which is the basic ingredient for noncommutative generalization of riemannian geometry [8]. In fact, the operator T gives the unit length scale of the manifold M and the infinitesimal line element ds in riemannian geometry is identified with the operator $1/|T|$. Another significance of the operator T is that its homotopy class represents the K-homology class of the manifold, which will be discussed in section 4.

Let us consider the triple $(\mathcal{H}, \widehat{\mathcal{A}}, T)$, where $\widehat{\mathcal{A}}$ is a C^*-algebra generated by Φ^{μ} acting on the Chan-Paton Hilbert space $\mathcal{H}$, and T is the (unbounded) tachyon operator, which is a self-adjoint operator on $\mathcal{H}$. We assume here that $\widehat{\mathcal{A}}$ is unital, i.e. $\widehat{\mathcal{A}} \ni \mathrm{id}_{\mathcal{H}}$, for simplicity. For the commutative case, this means that we consider the C^*-algebra $\widehat{\mathcal{A}} = C_0(M) = C(M)$ with compact space M. Note that if the topological space M is non-compact, $C_0(M)$ is not unital, since the constant function with value 1 is not an element of $C_0(M)$.

Let us consider here the following conditions.

$$(T-\lambda)^{-1} \in \mathbf{K}(\mathcal{H}) \quad \text{for } {}^{\forall}\lambda \notin \mathbb{R}, \qquad [\widehat{a}, T] \in \mathbf{B}(\mathcal{H}) \quad \text{for } {}^{\forall}\widehat{a} \in \widehat{\mathcal{A}}. \tag{3.1}$$

The triple $(\mathcal{H}, \widehat{\mathcal{A}}, T)$ satisfying these conditions is called a spectral triple [8]. (See also [27, 28].) The former condition in (3.1) means that T has a real discrete spectrum made of eigenvalues $\{\lambda_n \in \mathbb{R}\}$ with finite multiplicity such that $|\lambda_n| \to \infty$ as $n \to \infty$. This is what we expect from the finiteness of the tachyon potential, as we explained in section 2.1. The latter one in (3.1) is nothing but the third condition in (2.4), which is required from the finiteness of the tachyon kinetic term. Therefore the triple $(\mathcal{H}, \widehat{\mathcal{A}}, T)$ defined by a configuration of the K-matrix theory makes a spectral triple.

In general, we require a regularity hypothesis on spectral triples $(\mathcal{H}, \widehat{\mathcal{A}}, T)$ by replacing $\widehat{\mathcal{A}}$ with a dense involutive subalgebra of the C^*-algebra $\widehat{\mathcal{A}}$.[3] For example, we take $\widehat{\mathcal{A}} = C^\infty(M)$ instead of $C(M)$ for the commutative cases.

A basic example of the spectral triple, which is called the canonical triple, is given by $(\mathcal{H}, \widehat{\mathcal{A}}, T) = (L^2(M,S), C^\infty(M), D)$. Here M is a closed riemannian spin manifold,[4] $L^2(M,S)$ is the Hilbert space of square integrable sections of the spinor bundle on M, and D is the Dirac operator associated with the Levi-Civita connection of the metric. This is essentially nothing but the D-brane configurations considered in section 2.3 for the case $M = \mathbb{R}^{2m+1}$, though we assumed M to be compact and could be curved here. Here, we take $u = 1$ for the normalization of the tachyon in (2.19), so that $\| [\Phi^\mu, T] \| = 1$. As explained in the previous subsection, a point x of M is given by the character ϕ_x of $\widehat{\mathcal{A}} = C(M)$, and $x^\mu = \phi_x(\Phi^\mu)$ for the flat $M = \mathbb{R}^n$ case. Therefore the distance between two points $x_1, x_2 \in \mathbb{R}^n$ is given by $|\vec{x}_1 - \vec{x}_2| = |\phi_{x_1}(\vec{\Phi}) - \phi_{x_2}(\vec{\Phi})|$, where $\vec{x}_1 = (x_1^1, \ldots, x_1^n)$ etc. Then, it is not hard to imagine that this can be generalized as

$$d(\phi_1, \phi_2) = \sup_{a \in \widehat{\mathcal{A}}} \left\{ |\phi_1(a) - \phi_2(a)| \;\middle|\; \| [T, a] \| \leq 1 \right\}, \tag{3.2}$$

where ϕ_i $(i = 1,2)$ are linear functions $\phi_i : \widehat{\mathcal{A}} \to \mathbb{C}$ such that $\phi_i(a^* a) \geq 0$ for ${}^\forall a \in \widehat{\mathcal{A}}$ and normalized as $\phi_i(1) = 1$. Such functions as ϕ_i are called states and the distance $d(\phi_1, \phi_2)$ between two states in an arbitrary spectral triple is defined by this formula. It is known that this agrees with the geodesic distance between two points for the canonical triples, when we take ϕ_i as characters that represent the two points. (See for example [8, 28]).

In this way, the operator T carries information about the metric on the world-volume of the D-brane. More explicitly, for the canonical triple, the asymptotic expansion of the heat kernel of T^2 at small t is known as [29]

$$\mathrm{Tr}_{\mathcal{H}} \left(e^{-t\,T^2} \right) \sim \frac{2^{[n/2]}}{(4\pi t)^{n/2}} \int_M d^n x \sqrt{g} \left(1 + \frac{t}{12} R + O(t^2) \right), \tag{3.3}$$

from which we can measure the volume of the world-volume, integral of the mean curvature and so on. Note that the first term in the expansion is used in (2.23) to derive the CS-term of the D-brane.

Note that the metric defined in (3.2) is, in general, different from the usual metric of the D-brane induced from the background metric via the embedding, since the metric defined by the tachyon operator depends on the scale of the tachyon condensation. Namely, the unit length is defined by the scale of the tachyon condensation. Anyway, the action of the higher dimensional D-brane represented by the spectral triple $(\mathcal{H}, \widehat{\mathcal{A}}, T)$, should be written covariantly using the world-volume metric defined by the tachyon, which is analogous to the covariant Polyakov action in string theory.

Let us next explain the fact that information of the dimension of the D-brane world-volume is also hidden in the spectrum of T. As we can see in the expansion (3.3), the

[3] This subalgebra is generated by elements $\widehat{a} \in \widehat{\mathcal{A}}$ such that $\widehat{a}$ and $[T, \widehat{a}]$ are in the domain of δ^k, where $\delta(f) = [|T|, f]$ is the derivation of the operator $f \in \mathbf{B}(\mathcal{H})$.

[4] One can also construct a canonical triple over a spin^c manifold, by adding a U(1) gauge connection.

dimension n of the space M can be read from the power of t in the right hand side. More generally, the notion of dimension of a spectral triple is replaced by dimension spectrum which is a subset $\Sigma \subset \mathbb{C}$ of the singularities of the analytic function[5] [30, 8]

$$\zeta_T(z) = \mathrm{Tr}_{\mathcal{H}}(|T|^{-z}) \,. \tag{3.4}$$

It is easy to show that it gives the dimension n for the expansion (3.3), as expected, using the Mellin transform

$$\mathrm{Tr}_{\mathcal{H}}(|T|^{-z}) = \frac{1}{\Gamma(z/2)} \int_0^\infty t^{z/2-1}\, \mathrm{Tr}_{\mathcal{H}}(e^{-tT^2}) dt \,. \tag{3.5}$$

Interestingly, there are some examples that the dimension defined above does not take an integer value [8, 31]. It would be interesting if a fractal D-brane is realized in the K-matrix theory.

Now we discuss the diffeomorphism of the spectral triples. In [8], the diffeomorphism of the geometry represented by the spectral triple and the local gauge transformation on it are discussed. Then in the K-matrix theory, these may be realized in the spectral triples which represent the world-volume of the D-branes. Actually, a subset of the unitary operators in $\mathbf{B}(\mathcal{H})$ can be interpreted as {local gauge transf. $\times$ diffeo.} in the explicit examples of D-brane configurations with commutative world-volumes. Let us explain this below.

A configuration of curved N D$(2m)$-branes with the U(N) gauge field is given by

$$\begin{aligned} \Phi^\mu &= f^\mu(\hat{x}^i) \qquad (i = 0, \ldots, 2m) \\ T &= \frac{1}{2}\left\{\gamma^a e^i_{\ a}(\hat{x}), \left(\hat{p}_i + w_i^{ab}(\hat{x})\gamma_{ab} + A_i(\hat{x})\right)\right\}, \end{aligned} \tag{3.6}$$

where $[\hat{x}^i, \hat{x}^j] = 0$, $[\hat{x}^i, \hat{p}_j] = i\delta^i_j$ and $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$. Here we think that $A_i(\hat{x})$ is the U(N) gauge field, $f^\mu(\hat{x})$ is the embedding function, $e^i_{\ a}(\hat{x})$ are vielbein and $w_i^{ab}(\hat{x})$ is the spin connection constructed from them. Remember that the configurations of the theory should be physically identified with each other by the unitary transformations, since they are the gauge transformation of the underlying non-BPS D-instantons. However it is difficult to obtain the usual geometric picture for the transformed configurations, if Φ^μ in the transformed configuration depend on $\hat{p}^i$, γ^i or $N \times N$ matrices. Then in order to interpret the transformations in terms of the usual geometric picture, we will consider the unitary operators $u = e^{iH}$, where

$$H = \lambda(\hat{x}, \gamma^i) + \frac{1}{2}\{\hat{p}_i, \epsilon^i(\hat{x})\} \,, \tag{3.7}$$

ϵ^i is an identity as an $N \times N$ matrix, and λ is an $N \times N$ matrix valued function.

Since these operators form a subgroup and the action of the K-matrix theory is invariant under these transformations, we can in principle obtain an action for the D$(2m)$-branes, which is invariant under these transformations, by evaluating the action of the K-matrix theory including all the fluctuations around the configuration (3.6). However, the fields on

[5] More precisely, the dimension spectrum is the singularities of $\zeta_b(z) = \mathrm{Tr}_{\mathcal{H}}(b|T|^{-z})$ where b is an element of the algebra generated by $\delta^k(\hat{a})$, $\delta^k([T, \hat{a}])$ with $\hat{a} \in \hat{\mathcal{A}}$.

the N D($2m$)-branes in (3.6) do not include all the fluctuations. Indeed, there are the n-form fields associated to the $\gamma^{i_1}\cdots\gamma^{i_n}$, for example, and the transformation using $\lambda(\hat{x},\gamma^i)$ includes the n-form gauge transformation.

Here we consider the fields in (3.6) only and the transformations generated by (3.7) with $\lambda = \lambda(\hat{x})$ because these transformations form a subgroup and act consistently on these fields. Then the transformation using $u = \exp(i\lambda(\hat{x}))$ corresponds to the local gauge transformation since $u\Phi^\mu u^{-1} = \Phi^\mu$ and $uTu^{-1} = \gamma^a e^i_{\ a}(\hat{x})\left(\hat{p}_i + w_i(\hat{x}) + A^\lambda_i(\hat{x})\right)$, where $A^\lambda_i = uA_iu^{-1} - iu\partial_i u^{-1}$. The transformation using $u_d = \exp(i\frac{1}{2}\{\hat{p}_i,\epsilon^i(\hat{x})\})$ corresponds to the diffeomorphism of the world-volume of the D($2m$)-branes. For simplicity we assume that ϵ^i is infinitesimal. Then we can verify that $u_d\Phi^\mu u_d^{-1} = f^\mu(\hat{y}(\hat{x}))$ and $u_dTu_d^{-1} = \frac{1}{2}\{\gamma^a e'^i_{\ a}(\hat{x}), (\hat{p}_i + w'^{ab}_i(\hat{x})\gamma_{ab} + A'_i(\hat{x}))\}$, where we set $\hat{y}^i = \hat{x}^i + \epsilon^i(\hat{x})$ and

$$e'^i_{\ a} = \frac{\partial\hat{x}^i}{\partial\hat{y}^j}e^j_{\ a}(\hat{y}(\hat{x}))\,, \quad \hat{p}_i = -i\frac{\partial}{\partial\hat{x}^i}\,, \quad w'^{ab}_i = \frac{\partial\hat{y}^j}{\partial\hat{x}^i}w^{ab}_j(\hat{y}(\hat{x}))\,, \quad A'_i = \frac{\partial\hat{y}^j}{\partial\hat{x}^i}A_j(\hat{y}(\hat{x}))\,. \tag{3.8}$$

Therefore the world-volume theory of the D($2m$)-branes, which is the K-matrix action evaluated by (3.6), has the invariance under the diffeomorphism and local gauge transformation. Indeed, the large metric expansion of the K-matrix action is a Polyakov type action

$$S \sim \int dx^{2m+1}\sqrt{g}\left(1 + 2\log 2G_{\mu\nu}\partial_i f^\mu\partial_j f^\nu g^{ij} + \cdots\right), \tag{3.9}$$

where $g^{ij} = e^i_{\ a}\eta^{ab}e^j_{\ b}$ is the world-volume metric and $G_{\mu\nu} = \eta_{\mu\nu}$ is the background metric.

In the case of noncommutative D-brane configuration, we may also be able to give an interpretation to the unitary transformation as {local gauge transf. $\times$ diffeo.} as above.

3.3 Embedding of D-branes

The spectral triples considered in the previous subsection represent the geometry of the world-volume of D-branes, and we did not specify the space-time manifold in which the D-branes are embedded. In this subsection, we fix a space-time manifold X, and explain how to describe D-branes embedded in X in the algebraic description. We will use this set up for the classification of D-branes in the next section.

Let us recall the correspondence between the category of topological spaces (locally compact Hausdorff spaces) and the category of algebras (C^*-algebra). (See for example [27].)

Topology	$\leftrightarrow$	**Algebra**	:	commutative case
topological space X	$\leftrightarrow$	C^*-algebra $\mathcal{A}$	:	$C_0(X)$
compact	$\leftrightarrow$	unital	:	$C_0(X) = C(X) \ni 1$
proper map $\varphi : M \to X$	$\leftrightarrow$	$*$-homomorphism	:	$\varphi^* : C_0(X) \to C_0(M)$
homeomorphism	$\leftrightarrow$	automorphism	:	
open subset $U \subset X$	$\leftrightarrow$	ideal	:	J_{X-U}
closed subset $V \subset X$	$\leftrightarrow$	quotient algebra	:	$C_0(X)/J_V$

Table 1: The correspondence between topological spaces and C^*-algebras.

Here the ideal J_V associated with a closed subset $V \subset X$ is defined as $J_V = \{f \in C_0(X) \mid f|_V = 0\}$. The proper map φ is a continuous map such that the inverse image $\varphi^{-1}(K)$ of any compact subset K in X is compact. In other words, roughly speaking, φ maps infinity to infinity. Note that $\varphi^* f = f \circ \varphi$ for a function $f \in C_0(X)$ may not vanish at infinity, if φ is not a proper map.

Let us consider a D-brane world-volume M embedded in the space-time manifold X. Let $\mathcal{A} = C_0(X)$ and $\widehat{\mathcal{A}} = C_0(M)$ be the algebra corresponding to X and M respectively. Since M is a closed subset of X and does not have a boundary except infinity, the inclusion map $i : M \to X$ should be a proper map. Therefore the inclusion map induces a $*$-homomorphism $i^* : \mathcal{A} \to \widehat{\mathcal{A}} \simeq \mathcal{A}/J_M$. The ideal J_M is the kernel of i^* by definition. The generalization to the noncommutative cases is straightforward. Let $\mathcal{A}$ be the C^*-algebra that corresponds to the space-time manifold, which could be noncommutative. In order to obtain an algebra $\widehat{\mathcal{A}}$ corresponding to the world-volume of the D-brane embedded in the space-time, we choose a *-homomorphism $\phi : \mathcal{A} \to \mathbf{B}(\mathcal{H})$ and set $\widehat{\mathcal{A}} = \mathrm{Image}\,\phi \cong \mathcal{A}/\ker\phi$. For example, suppose that the space-time algebra is $\mathcal{A} = C(X)$, where X is a compact subset of $\mathbb{R}^N$, and a $*$-homomorphism $\phi : \mathcal{A} \to \mathbf{B}(\mathcal{H})$ is given, $\widehat{\mathcal{A}}$ is defined as the algebra generated by $\Phi^\mu = \phi(x^\mu)$, where $(x^0, \ldots, x^{N-1})$ is the coordinate of $\mathbb{R}^N$.

As emphasized in [32], we do not apriori have the notion of space-time manifold in matrix theory. As an example, consider the IIA K-matrix theory formulated in the flat $\mathbb{R}^{10}$ background. The eigenvalues of the scalar field Φ^μ represent the positions of the non-BPS D-instantons in x^μ direction of the $\mathbb{R}^{10}$. However, there are configurations that Φ^μ are mutually noncommutative and they cannot be simultaneously diagonalized. In such cases, we cannot say that the non-BPS D-instantons live in the $\mathbb{R}^{10}$ space-time.

But, it is still interesting to consider the possible configurations of D-branes embedded in a fixed space-time manifold using the framework of the matrix theory. For instance, if we are only interested in the commutative D-branes embedded in $\mathbb{R}^{10}$, it is reasonable to fix the space-time algebra as $\mathcal{A} = C_0(\mathbb{R}^{10})$ and consider only the D-branes represented by the algebra $\widehat{\mathcal{A}} = \mathrm{Image}\,\phi$ for some $*$-homomorphism $\phi : \mathcal{A} \to \mathbf{B}(\mathcal{H})$. Note that, in this case, we can never obtain D-brane configurations with noncommutative world-volume algebra $\widehat{\mathcal{A}}$ as the image of ϕ, since $\mathcal{A} = C_0(\mathbb{R}^{10})$ is commutative and the map ϕ is homomorphism.

In the next section, we will first fix a C^*-algebra $\mathcal{A}$, which we call the space-time algebra, and classify the stable D-brane configurations which are embedded in the space-time represented by $\mathcal{A}$. Then, D-branes embedded in the space-time are represented by the spectral triples $(\mathcal{H}, \widehat{\mathcal{A}}, T)$, where $\widehat{\mathcal{A}}$ is given by $\widehat{\mathcal{A}} = \mathrm{Image}\,\phi$, using a $*$-homomorphism $\phi : \mathcal{A} \to \mathbf{B}(\mathcal{H})$. In other words, such D-branes are obtained by a triple $(\mathcal{H}, \phi, T)$, which is called an (unbounded) Fredholm module. Therefore, the classification of D-brane configurations are obtained by classifying the Fredholm modules, which we will demonstrate in the next section.

4. D-branes and K-homology

4.1 Classification of D-brane configurations

The D-brane charge is classified by K-theory group $K^1(X)$ ($K^0(X)$) in type-IIA (IIB) string theory [2, 4]. So, at first sight, one might think that the D-brane configurations in the K-matrix theory should be naturally classified by the algebraic K-theory $K_1(\mathcal{A})$ ($K_0(\mathcal{A})$), since they are isomorphic to the topological K-theory $K^i(X) = K_i(C(X))$ when $\mathcal{A} = C(X)$. However, it turns out that this is *not* the correct answer. Let us explain this fact shortly.[6]

The charge of D-branes is usually defined by the behavior of RR-fields. Therefore, it should be classified by cohomology theory. Actually, K-theory is a kind of refined cohomology theory. In particular, it behaves as a contravariant functor from the category of topological spaces to the category of abelian groups. This means that a diffeomorphism $\phi : X \to X'$ induces a pull-back map

$$\phi^* : K^i(X') \to K^i(X)\,. \tag{4.1}$$

On the other hand, when we construct D-branes in the K-matrix theory with $\mathcal{A} = C(X)$, the D-brane solutions represent cycles of X which correspond to the world-volume of the D-branes. Hence, they should be classified by homology theory, which is Poincaré dual to the cohomology theory and transforms covariantly under the diffeomorphism ϕ.

There is a group called K-homology $K_i(X) = K^i(C(X))$, which is dual to the K-theory group $K^i(X) = K_i(C(X))$ in the sense that it has a natural pairing with the K-theory group,

$$K^i(X) \times K_i(X) \to \mathbb{Z}\,. \tag{4.2}$$

Accordingly, the K-homology $K_i(X)$ is a homological object, which is preferable to our purpose. In the next subsection, we claim that the K-homology $K^1(\mathcal{A})$ ($K^0(\mathcal{A})$) is the group which classifies the D-brane configurations in the type-IIA (IIB) K-matrix theory. (See also [34, 33, 35].)

4.2 Analytic K-homology

There are several ways to define the K-homology. (See, for example, [36, 37, 39].) One of them is the definition of the K-homology using Fredholm operators. We can also define it in terms of manifolds and vector bundles when the algebra $\mathcal{A}$ is commutative. As we will see in this subsection, the former have a direct physical interpretation in the K-matrix theory, since our formulation is based on the operator algebra. The latter topological approach is useful to relate the elements of K-homology to world-volume configurations of D-branes, and will be discussed in the next subsection.

First we fix the space-time C^*-algebra $\mathcal{A}$. We assume $\mathcal{A}$ to be unital for simplicity. As explained in section 3.3, the D-branes embedded in the fixed space-time algebra $\mathcal{A}$ are obtained by the Fredholm modules. A Fredholm module over an algebra $\mathcal{A}$ is a triple $(\mathcal{H}, \phi, F)$, where

- $\mathcal{H}$ is a separable Hilbert space,
- $\phi : \mathcal{A} \to \mathbf{B}(\mathcal{H})$ is a $*$-homomorphism,

[6] A similar argument is given in [33].

- F is a self adjoint operator in $\mathbf{B}(\mathcal{H})$, which satisfies

$$F^2 - 1 \in \mathbf{K}(\mathcal{H})\,, \quad [F, \phi(a)] \in \mathbf{K}(\mathcal{H}) \qquad \text{for } {}^\forall a \in \mathcal{A}\,. \tag{4.3}$$

As explained in section 3.3, a Fredholm module $(\mathcal{H}, \phi, F)$ describes a configurations of the IIA K-matrix theory. $\mathcal{H}$ is identified as a space of Chan-Paton indices of the non-BPS D-instantons, the $*$-homomorphism ϕ specifies the world-volume $\widehat{\mathcal{A}} = \text{Image}\ \phi$ of the D-branes embedded in the space-time algebra $\mathcal{A}$, and the operator F is the normalized tachyon field T_b. The condition (4.3) is nothing but the condition (2.7), which is required from the finiteness of the action.

We also define a degenerate Fredholm module which is a Fredholm module satisfying $F^2 - 1 = [F, \phi(a)] = 0$. This corresponds to virtual non-BPS D-instantons that would be annihilated by the tachyon condensation. The sum of two Fredholm modules $(\mathcal{H}_i, \phi_i, F_i)$ $(i = 0, 1)$ are defined by the direct sum $(\mathcal{H}_0 \oplus \mathcal{H}_1, \phi_0 \oplus \phi_1, F_0 \oplus F_1)$.

Two Fredholm modules $(\mathcal{H}_i, \phi_i, F_i)$ $(i = 0, 1)$ are said to be unitary equivalent when there is a unitary operator in $\mathbf{B}(\mathcal{H}_0, \mathcal{H}_1)$ intertwining ϕ_i and F_i. They are operator homotopic if $\mathcal{H}_0 = \mathcal{H}_1$, $\phi_0 = \phi_1$ and there is a norm continuous path between F_0 and F_1. We define an equivalence relation $\sim$ on Fredholm modules generated by unitary equivalence, addition of degenerate elements and operator homotopy of $(\mathcal{H}, \phi, F)$. Then K-homology $K^1(\mathcal{A})$ is defined as the set of equivalence classes of the Fredholm modules under the equivalence relation $\sim$.

These equivalence relations have nice physical interpretations. The unitary equivalence is nothing but the gauge equivalence, addition of the degenerate elements means the addition of non-BPS D-instantons that would be annihilated by the tachyon condensation, and the operator homotopy is just a continuous deformation of the tachyon configuration. One could also consider the continuous deformation of $\mathcal{H}$ and ϕ, though it is known that the equivalence class is unchanged [37]. Therefore the equivalence relations considered above is physically enough for the classification of the configurations.

The K-homology which classifies the D-brane configurations in the IIB K-matrix theory is $K^0(\mathcal{A})$. It can be defined in a similar way. In this case, a Fredholm module over an algebra $\mathcal{A}$ is a 5-tuple $(\mathcal{H}^{(0)}, \mathcal{H}^{(1)}, \phi_0, \phi_1, F)$, where

- $\mathcal{H}^{(i)}$ are separable Hilbert spaces $(i = 0, 1)$,
- $\phi_i : \mathcal{A} \to \mathbf{B}(\mathcal{H}^{(i)})$ are $*$-homomorphisms $(i = 0, 1)$,
- F is an operator in $\mathbf{B}(\mathcal{H}^{(0)}, \mathcal{H}^{(1)})$, which satisfies

$$F^*F - 1 \in \mathbf{K}(\mathcal{H}^{(0)})\,, \qquad FF^* - 1 \in \mathbf{K}(\mathcal{H}^{(1)})\,, \tag{4.4}$$

$$F\phi_0(a) - \phi_1(a)F \in \mathbf{K}(\mathcal{H}^{(0)}, \mathcal{H}^{(1)}) \qquad \text{for } {}^\forall a \in \mathcal{A}\,. \tag{4.5}$$

This Fredholm module $(\mathcal{H}^{(0)}, \mathcal{H}^{(1)}, \phi_0, \phi_1, F)$ describes the configurations of the IIB K-matrix theory in an analogous way as above. $\mathcal{H}^{(0)}$ and $\mathcal{H}^{(1)}$ corresponds to the Chan-Paton indices of D-instantons and anti D-instantons, respectively. The $*$-homomorphisms ϕ_0 and ϕ_1 is used to obtain the configurations that D-instantons and anti D-instantons are settled inside the space-time manifold in the same way as explained in section 3.3 for non-BPS D-instantons. F is again the normalized tachyon field T_b. Then, (4.4) and (4.5) are the conditions corresponding to (2.8) and (2.9), respectively.

The degenerate Fredholm module is defined as the 5-tuple $(\mathcal{H}^{(0)}, \mathcal{H}^{(1)}, \phi_0, \phi_1, F)$ with $\mathcal{H}^{(0)} = \mathcal{H}^{(1)}$, $\phi_0 = \phi_1$ and $F = \mathrm{id}_{\mathcal{H}}$ is the identity operator of $\mathcal{H} \equiv \mathcal{H}^{(0)} = \mathcal{H}^{(1)}$. The K-homology $K^0(\mathcal{A})$ is defined as the set of equivalence classes of the Fredholm module. The equivalence relations of the Fredholm modules are again generated by unitary equivalence, addition of degenerate elements and the operator homotopy.

We can also rewrite these conditions in terms of the $\mathbb{Z}_2$ graded Hilbert space $\widehat{\mathcal{H}} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)}$. The conditions (4.4) and (4.5) can be expressed as

$$\widehat{F}^2 - 1 \in \mathbf{K}(\widehat{\mathcal{H}}), \quad [\widehat{\phi}(a), \widehat{F}] \in \mathbf{K}(\widehat{\mathcal{H}}), \qquad \text{for } {}^\forall a \in \mathcal{A}, \tag{4.6}$$

where $\widehat{\phi}(a) = \mathrm{diag}(\phi_0(a), \phi_1(a))$ and $\widehat{F} = \begin{pmatrix} & F^* \\ F & \end{pmatrix}$.

When $\mathcal{A} = C(X)$, where X is a compact manifold, there is a surjective map from $K^0(C(X)) = K_0(X)$ to $\mathbb{Z}$,

$$\mathrm{Index} : K_0(X) \to \mathbb{Z}\,. \tag{4.7}$$

This map is defined by taking the index of the Fredholm operator F,

$$\mathrm{Index}((\mathcal{H}^{(0)}, \mathcal{H}^{(1)}, \phi_0, \phi_1, F)) \equiv \mathrm{Index}\, F\,. \tag{4.8}$$

The index of a Fredholm operator is invariant under operator homotopy and the map (4.7) is well defined. Recall that F is the tachyon field of the IIB K-matrix theory and gives a map from D-instanton Chan-Paton Hilbert space $\mathcal{H}^{(0)}$ to anti D-instanton Chan-Paton Hilbert space $\mathcal{H}^{(1)}$. The basis of $\mathrm{Ker}\, F$ and $\mathrm{Coker}\, F$ correspond to the Chan-Paton indices for the D-instantons and anti D-instantons which are not annihilated by the tachyon condensation, respectively. Therefore the integer $\mathrm{Index}\, F = \mathrm{Ker}\, F - \mathrm{Coker}\, F$ is interpreted as the total number of D-instantons. Note that we can realize the configurations with any numbers of D-instantons in the K-matrix theory. This is one of the advantage of the K-matrix theory in contrast to the other matrix theories.

Let us examine $\mathcal{A} = C(S^n)$ case as a basic example. K-homology groups for this algebra are known as

$$K_0(S^n) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & (n : \text{even}) \\ \mathbb{Z} & (n : \text{odd})\,, \end{cases} \qquad K_1(S^n) = \begin{cases} 0 & (n : \text{even}) \\ \mathbb{Z} & (n : \text{odd})\,. \end{cases} \tag{4.9}$$

These results are consistent with what we expect from the homology group of S^n. S^n has non-trivial homology for $H_0(S^n) = \mathbb{Z}$ and $H_n(S^n) = \mathbb{Z}$, and hence the only topologically non-trivial D-branes wrapped on S^n are expected to be D-instantons and D$(n-1)$-branes. Here n should be odd (even) in type-IIA (IIB) string theory to obtain a stable D-branes wrapped on S^n. Since we always have the $\mathbb{Z}$ factor corresponding to the D-instantons, it is convenient to consider the reduced K-homology group, defined as the kernel of the index map (4.7), or equivalently replacing $C(S^n)$ with $\mathcal{A} = C_0(\mathbb{R}^n)$.

$$K_0(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & (n : \text{even}) \\ 0 & (n : \text{odd})\,, \end{cases} \qquad K_1(\mathbb{R}^n) = \begin{cases} 0 & (n : \text{even}) \\ \mathbb{Z} & (n : \text{odd})\,. \end{cases} \tag{4.10}$$

Here $K_i(\mathbb{R}^n)$ denote $K^i(C_0(\mathbb{R}^n))$. In this case, since the D-instantons can be kicked off to infinity, the $\mathbb{Z}$ factor corresponding to the D-instanton number is dropped. Therefore we could say that the flat Dp-brane is classified by $K_1(\mathbb{R}^{p+1})$ ($K_0(\mathbb{R}^{p+1})$) in the IIA (IIB) K-matrix theory.

Note that in the K-theory classification of D-brane charges [2], $K^i(\mathbb{R}^{9-p})$ is the group[7] which classifies the charge of the flat Dp-branes considered above. The space $\mathbb{R}^{9-p}$ is the dual of $\mathbb{R}^{p+1}$ in the space-time $\mathbb{R}^{10}$. This comes from the fact that K-theory of X classifies the D-brane charge defined by RR-fields on X, while K-homology of X classifies the world-volume of the D-brane embedded in the space-time manifold X, and they are related by Poincaré duality.

In general, for a n-dimensional compact manifold X, the K-theory groups and the K-homology groups are related by

$$K_i(X) \simeq K^{n-i}(X)\,, \tag{4.11}$$

where the subscript i and the superscript $n-i$ are understood modulo 2. This isomorphism is the K-theory lift of the Poincaré duality

$$H_i(X;\mathbb{Z}) \simeq H^{n-i}(X;\mathbb{Z})\,. \tag{4.12}$$

The isomorphism (4.11) is also reasonable from the physical point of view. For example, in type-IIA string theory, $K_1(X)$ classifies the D-brane constructed by non-BPS D-instanton system and $K^{n-1}(X)$ classifies the D-brane constructed by non-BPS D$(n-1)$-brane system when n is even, or D$(n-1)$-brane–anti D$(n-1)$-brane system when n is odd. The spectrum of the D-branes should not depend on how they are constructed, and hence $K_1(X)$ and $K^{n-1}(X)$ should be isomorphic. We will generalize this discussion in section 4.5.

4.3 Topological K-homology

When the algebra $\mathcal{A}$ is commutative, we have a topological definition of the K-homology, which is isomorphic to the analytic one given in the previous subsection. (See [36, 34, 39].) In this case, we can assume $\mathcal{A} = C_0(X)$ without any loss of generality. Here X can be any locally compact Hausdorff space, but, for simplicity, we assume that X is a closed (i.e. compact without boundaries) smooth manifold in this subsection.

A K-cycle on X is defined to be a triple (M, E, φ), where M is a compact Spin^c manifold without boundary, E is a complex vector bundle on M, and φ is a continuous map from M to X. Note that we don't require the manifold M to be connected, and the rank of E may be different on different connected components of M. Therefore, the disjoint union $(M_0, E_0, \varphi_0) \cup (M_1, E_1, \varphi_1)$ of two K-cycles (M_i, E_i, φ_i) $(i = 0, 1)$ is again a K-cycle.

The (topological) K-homology $K_*^{\mathrm{top}}(X) = K^*_{\mathrm{top}}(C(X))$ is the set of equivalence classes of the K-cycles. The equivalence relations are generated by the following (a)∼(c):

1. Bordism

[7] Here $K^i(\mathbb{R}^n)$ denotes the reduced K-theory group of S^n, $\widetilde{K}^i(S^n)$ [38].

$(M_0, E_0, \varphi_0) \sim (M_1, E_1, \varphi_1)$ if there exists a triple (W, E, φ), such that $(\partial W, E|_{\partial W}, \varphi|_{\partial W})$ is isomorphic to the disjoint union $(M_0, E_0, \varphi_0) \cup (-M_1, E_1, \varphi_1)$. Here W is a compact Spin^c manifold with boundary, E is a complex vector bundle on W, φ is a continuous map from W to X, and $-M_1$ denotes M_1 with the reversed Spin^c structure.

2. Direct sum

   $(M, E_1 \oplus E_2, \varphi) \sim (M, E_1, \varphi) \cup (M, E_2, \varphi)$

3. Vector bundle modification

   $(M, E, \varphi) \sim (\widehat{M}, \widehat{H} \otimes \rho^*(E), \varphi \circ \rho)$, where $\widehat{M}$ is a sphere bundle on M whose fiber S_p is an even dimensional sphere, ρ is the projection $\widehat{M} \to M$ and $\widehat{H}$ is a vector bundle on $\widehat{M}$, such that for each $p \in M$ the restriction of $\widehat{H}$ to $S_p = \rho^{-1}(p)$ is the generator of $\widetilde{K}(S_p) = \mathbb{Z}$. (See [36, section 10] for the explicit construction.)

The sum of two elements in the K-homology is defined by the disjoint union, and it can be shown that $K_*^{\rm top}(X)$ is an abelian group. $K_*^{\rm top}(X)$ is a direct sum of two subgroups $K_i^{\rm top}(X)$ $(i = 0, 1)$,

$$K_*^{\rm top}(X) = K_0^{\rm top}(X) \oplus K_1^{\rm top}(X)\,, \tag{4.13}$$

where $K_0^{\rm top}(X)$ $(K_1^{\rm top}(X))$ consists of the elements given by the K-cycles (M, E, φ) with each component of M even (odd) dimensional.

It is natural to interpret the K-cycle (M, E, φ) as the world-volume of the D-brane as proposed in [34]. M is interpreted as the world-volume of the D-brane with Chan-Paton bundle E on it and φ determines the embedding of the D-brane to the space-time X. The requirement that M is equipped with a Spin^c structure is consistent with the fact that D-branes cannot wrap on a cycle without any Spin^c structures [40, 41, 2, 42].

The topological K-homology $K_1^{\rm top}(X)$ $(K_0^{\rm top}(X))$ defined above nicely classifies the stable D-brane configurations in type-IIA (IIB) string theory. The equivalence relation (a) is the deformations of the world-volume of the D-brane together with the gauge bundle on it, the relation (b) represents the process of the gauge symmetry enhancement for coincident D-branes. The relation (c) is the descent relation of the D-branes, namely it means that we should identify a spherical D-brane with a non-trivial gauge bundle on it with a lower dimensional D-brane. Let us explain this fact in a little more detail. The Spin^c manifold $\widehat{M}$ and the vector bundle $\widehat{H}$ are constructed as follows. Let H be a Spin^c vector bundle on M with $2n$ dimensional fibers, and $B(H)$ be the unit ball bundle of H. The boundary of $B(H)$ is a unit sphere bundle $S(H)$ on M, whose fiber is $2n-1$ dimensional sphere. $\widehat{M}$ is defined by gluing two copies of $B(H)$, denoted by $B(H)_+$ and $B(H)_-$, by the identity map of $S(H)$. Thus, $\widehat{M}$ is a sphere bundle on M with $2n$ dimensional sphere as its fiber, and is also a Spin^c manifold. $B(H)_+$ and $B(H)_-$ are regarded as the world-volume of a D$(2n+p)$-brane and an anti D$(2n+p)$-brane, respectively, where $p = \dim M - 1$. Gluing them together, $\widehat{M}$ can be thought of as the world-volume of a spherical D$(2n+p)$-brane, wrapped on $2n$ dimensional sphere. Since the fiber of the Spin^c vector bundle H is even dimensional, we can define two spinor bundles $H_\pm$, labeled by chirality, on M associated to H. Let $S_\pm$

be the pull-backs of $H_\pm$ to H, using the projection $H \to M$. We associate S_+ and S_- as the Chan-Paton vector bundle on the D$(2n+p)$-brane and the anti D$(2n+p)$-brane, respectively. We restrict the base H of $S_\pm$ to $B(H)_\pm$ and denote them by $S_\pm|_{B(H)_\pm}$. The vector bundle $\widehat{H}$ on $\widehat{M}$ is constructed by gluing $S_+|_{B(H)_+}$ and $S_-|_{B(H)_-}$ by the transition function g on $S(H)$. The transition function g is defined by

$$g(x, v) = v_\mu \gamma^\mu \,, \tag{4.14}$$

where $x \in M$, v is unit vector of the $2n$ dimensional vector space, which is the element of the fiber of $S(H)$ at the point x, and γ^μ is the SO$(2n)$ gamma matrices restricted on the space of positive chirality spinors. The transition function is interpreted as the tachyon field created by the open string stretched between the D$(2n+p)$-brane and the anti D$(2n+p)$-brane, and this tachyon configuration (4.14) induces a unit Dp-brane charge [2]. Therefore, this configuration should be physically identified with Dp-brane world-volume characterized by (M, E, φ). This is the physical meaning of the equivalence relation (c).

As mentioned above, one can show that the topological K-homology is isomorphic to the analytic K-homology, which we described in the previous subsection. The isomorphism

$$\mu_i : K_i^{\mathrm{top}}(X) \overset{\sim}{\to} K_i(X) \qquad (i = 0, 1) \tag{4.15}$$

is given as follows. To be specific, we will explain the $i = 1$ case. Let (M, E, φ) be an element of $K_1^{\mathrm{top}}(X)$. Since M is an odd dimensional closed Spin^c manifold, we can define a spin bundle S associated to the spinor representation of the Spin^c group. $\Gamma(S \otimes E)$ denotes the space of smooth sections of the vector bundle $S \otimes E$ on M. We can define a Dirac operator D on $\Gamma(S \otimes E)$ by choosing a connection on the bundle $S \otimes E$ as usual. $\Gamma(S \otimes E)$ is equipped with an inner product, and the Hilbert space $\mathcal{H} = L^2(M, S \otimes E)$ is defined by the completion of $\Gamma(S \otimes E)$ with respect to the inner product. The Dirac operator D can be thought of as an (unbounded) operator on $\mathcal{H}$. The representation $\phi : C(X) \to \mathbf{B}(\mathcal{H})$ is defined by the multiplication of the function $\phi(f) \equiv f \circ \varphi$ for each function $f \in C(X)$. Thus, we have obtained an unbounded Fredholm module $(\mathcal{H}, \phi, D)$. It can be shown that this defines an element of the K-homology $K_1(X)$ irrespective of the choice of the connection on $S \otimes E$, and furthermore, it gives a well-defined map from $K_1^{\mathrm{top}}(X)$ to $K_1(X)$, which turns out to be an isomorphism.

As we have seen in section 2.3, the tachyon operator for the D-brane solution in [22] is nothing but a Dirac operator acting on a spin bundle on $M = \mathbb{R}^{2m+1}$. Therefore this D-brane configuration in the K-matrix theory corresponds to the element of topological K-homology with $M = \mathbb{R}^{2m+1}$ and $E = I$ (trivial line bundle), which is interpreted as a D$(2m)$-brane extending along $\mathbb{R}^{2m+1}$ with trivial Chan-Paton bundle. This is exactly what we expect from the calculation of the tension and the Chern-Simons term [22]. The isomorphism (4.15) suggests that we can always obtain a world-volume interpretation for each configuration in the K-matrix theory. Since we have a clear geometrical interpretation of the D-brane configurations in the topological K-homology, it provides a convincing evidence for our proposal that the D-brane configurations in the K-matrix theory is classified by the (analytic) K-homology.

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4.4 Chern character and Chern-Simons terms

In the K-theory description of D-branes, a D-brane is constructed as the gauge configuration on non-BPS D9-brane system (type IIA) or D9-$\overline{\text{D9}}$ system (type IIB), and the D-brane charge is classified by the K-theory groups $K^*(X)$ [2, 4]. But when we are not interested in the torsion part of the K-theory groups, it is enough to use the cohomology group $H^*(X)$ to classify the D-brane charges, and we can read them from the Chern-Simons terms.

In order to write down the Chern-Simons term, the Chern character plays a crucial role [43]. Namely, the Chern character induces isomorphisms

$$\mathrm{ch} : K^0(X) \otimes \mathbb{Q} \xrightarrow{\sim} H^{\mathrm{even}}(X; \mathbb{Q}) \,, \tag{4.16}$$

$$\mathrm{ch} : K^1(X) \otimes \mathbb{Q} \xrightarrow{\sim} H^{\mathrm{odd}}(X; \mathbb{Q}) \,, \tag{4.17}$$

between the K-theory groups tensored by $\mathbb{Q}$ and the cohomology groups. Using these maps the Chern-Simons term for the world-volume theory of D9-branes (in the flat background) can be written as

$$S_{\mathrm{CS}} = \int_X C \wedge \mathrm{ch}(x) \,, \tag{4.18}$$

where C is the formal sum of RR-fields and x is an element of $K^*(X)$. The Chern character $\mathrm{ch}(x)$ can be explicitly written in terms of superconnections [44]. For example, in type-IIA string theory, the relevant superconnection is defined by

$$i\mathcal{A} = \begin{pmatrix} iA & T \\ T & iA \end{pmatrix} = iA \otimes I_2 + T \otimes \sigma^1 \,, \tag{4.19}$$

$$\mathcal{F} = d\mathcal{A} - i\mathcal{A}^2 = i \begin{pmatrix} F - T^2 & DT \\ DT & F - T^2 \end{pmatrix} \,, \tag{4.20}$$

where T is the tachyon and A is the gauge field on the world-volume of the non-BPS D9-brane. Using these variables, (4.18) becomes [15, 16]

$$S_{\mathrm{CS}} = \int_X C \wedge \mathrm{Tr}_N \left(\mathrm{Tr}_2 \left(\sigma^1 e^{\mathcal{F}} \right) \right) . \tag{4.21}$$

In this formula, one can easily show that $\mathrm{Tr}_N \, \mathrm{Tr}_2 \, \sigma^1 e^{\mathcal{F}}$ is a closed form, which ensures the invariance under the gauge transformation $C \to C + d\Lambda$.

What is the counterpart of this in the K-homology? There is a theorem analogous to (4.16) and (4.17), which claims that the K-homology group is isomorphic to the ordinary homology group if it is tensored by $\mathbb{Q}$ [36]:

$$\mathrm{ch.} : K_0(X) \otimes \mathbb{Q} \xrightarrow{\sim} H_{\mathrm{even}}(X; \mathbb{Q}) \,, \tag{4.22}$$

$$\mathrm{ch.} : K_1(X) \otimes \mathbb{Q} \xrightarrow{\sim} H_{\mathrm{odd}}(X; \mathbb{Q}) \,. \tag{4.23}$$

Here the element $(M, E, \varphi) \in K_*(X)$ of topological K-homology is mapped to

$$\mathrm{ch.}(M, E, \varphi) = \varphi_*(\mathrm{ch}(E) \cup \mathrm{Td}(TM) \cap [M]) \,. \tag{4.24}$$

Since the homology group in (4.22) and (4.23) classifies the world-volume cycle of the D-brane which corresponds to the element (M, E, φ) of the K-homology, the Chern-Simons term should be written by integrating C over the cycle (4.24). Hence we obtain

$$S_{\rm CS} = \int_M \varphi^* C \wedge {\rm ch}(E) \wedge {\rm Td}(TM)\,, \tag{4.25}$$

which agrees with the Chern-Simons term for a D-brane of world-volume M with Chan-Paton bundle E, dropping the factor that comes from the background curvature on X [45, 43]. This is again consistent with the interpretation that M is the world-volume of the brane and E is the Chan-Paton bundle on it.

On the other hand, we have an analytic description of the K-homology group. How can we define the Chern character in this formulation and relate it to the Chern-Simons terms? To answer this question, note that we can rewrite the Chern-Simons term for the IIA K-matrix theory (2.17) as

$$S_{\rm CS} = \sum_n \Psi_{2n+1}\left(C_{\mu_1\cdots\mu_n}, x^{\mu_1}, \ldots, x^{\mu_n}\right), \tag{4.26}$$

where

$$\begin{aligned}\Psi_{2n+1}(a^0, a^1, \ldots, a^{2n+1}) = &\int_{\sum s_i = 1, s_i \geq 0} ds_0 \cdots ds_{2n} \times \\ &\times {\rm Tr}_{\mathcal{H}}\left(\phi(a^0) e^{-s_0 T^2} [T, \phi(a^1)] e^{-s_1 T^2} \cdots [T, \phi(a^{2n+1})] e^{-s_{2n+1} T^2}\right),\end{aligned} \tag{4.27}$$

and $a^i \in \mathcal{A}$. $\Psi = (\Psi_{2n+1})$ is known as (odd) JLO cocycle [46] associated with the unbounded Fredholm module $(\mathcal{H}, \phi, T)$. This is a Chern character of the K-homology $K^1(\mathcal{A})$ that takes value in the entire cyclic cohomology $HE^*(\mathcal{A})$. (See [8].) In this case, the JLO cocycle has similar role as the usual Chern character or superconnection. The JLO formula (4.27) can also be used in the case that $\mathcal{A}$ is noncommutative. It seems to be quite natural that the CS-term for the D-brane in noncommutative manifold is also obtained from the Chern character of the K-homology. However, it is not clear in the above formula how to incorporate the Myers' terms [21] and the RR-fields which are not restricted to be constant.

4.5 KK-theory

In this subsection, we will leave the K-matrix theory for a while and deal with the classification of D-branes in a slightly more general context. Let us consider the field theory of higher dimensional D-branes as a fundamental theory instead of D-instantons, to be precise, the non-BPS Dp-brane system or Dp-$\overline{{\rm D}p}$ system in type-II string theory. As for the K-matrix theory, we can classify possible stable configurations of the theory and claim that in this case the appropriate group is KK-theory, which is a generalization of both K-theory and K-homology. We will also see that this classification is in fact equivalent to that of K-matrix theory in a simple example.

Let (A, B) be a pair of C^*-algebras.[8] The KK-group $KK(A, B)$ is an abelian group associated with (A, B), which is covariant in B and contravariant in A. Roughly speaking,

[8] Here we assume that both A and B are unital.

it is defined by equivalence classes of Hilbert (A, B)-bimodules, called Kasparov modules.[9] There is a natural abelian group structure on $KK(A, B)$ induced by the direct sum of Kasparov modules. More generally, $KK^n(A, B)$ is defined by $KK(A, B \otimes \mathbb{C}_n)$, where $\mathbb{C}_n$ is the complex Clifford algebra of $\mathbb{R}^n$.

Although there are various expressions for Kasparov modules, the Fredholm picture of $KK(A, B)$, in which the tachyon operator T carries almost all non-trivial information, is the appropriate one for us.

In precise, an odd Kasparov module in the Fredholm picture is defined by a triple $(\mathcal{H}_B, \phi, T)$, where

- $\mathcal{H}_B = B^\infty$ is a Hilbert space over B,
- $\phi : A \to \mathbf{B}(\mathcal{H}_B)$ is a $*$-homomorphism,
- T is a self-adjoint operator in $\mathbf{B}(\mathcal{H}_B)$ such that

$$T^2 - 1,\ [T, \phi(a)]\ \in \mathbf{K}(\mathcal{H}_B) = B \otimes \mathbf{K} \qquad \text{for } {}^\forall a \in A\,. \tag{4.28}$$

Note the similarity in the case of K-homology. Since $\mathcal{H}_B$ is a family of Hilbert spaces on the (possibly noncommutative) space B, a triple $(\mathcal{H}_B, \phi, T)$ is also a family of Fredholm modules on the space B. Of course, for $B = \mathbb{C}$ (one point space) it reduces to the (odd) Fredholm module described in section 4.2. The elements of $KK^1(A, B)$ are homotopy equivalence classes of odd Kasparov modules, with the similar equivalence relations as those used in the definition of K-homology in section 4.2.

An even Kasparov module $(\mathcal{H}_B^{(0)} \oplus \mathcal{H}_B^{(1)}, \phi^{(0)} \oplus \phi^{(1)}, \widehat{F})$ is defined by almost the same condition above, except that it has $\mathbb{Z}_2$ grading given by a standard self-adjoint involution operator γ. Namely, in the matrix form one has

$$\widehat{\mathcal{H}}_B = \begin{pmatrix} \mathcal{H}_B^{(0)} \\ \mathcal{H}_B^{(1)} \end{pmatrix}, \qquad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \widehat{\phi} = \begin{pmatrix} \phi^{(0)} & 0 \\ 0 & \phi^{(1)} \end{pmatrix}, \qquad \widehat{F} = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}. \tag{4.29}$$

The equivalence classes of even Kasparov modules define $KK(A, B) = KK^0(A, B)$. Note that odd Kasparov modules are also described by the matrix form as above with $\phi^{(0)} = \phi^{(1)}$ and $T = T^*$, which is equivalent to $\phi = I_2 \otimes \phi^{(0)}$ and $F = \sigma^1 \otimes T$, using generators $\{I_2, \sigma^1\}$ of $\mathbb{C}_1$. In fact, one can show that $KK^1(A, B)$ is isomorphic to $KK(A, B \otimes \mathbb{C}_1)$.

This set-up fits nicely to the classification of D-branes made from the non-BPS Dp-brane or Dp-$\overline{\mathrm{D}p}$ system with $p+1$ dimensional world-volume B. Stable D-brane configurations made out of the non-BPS Dp-brane and embedded in transverse space A are classified by $KK^1(A, B)$, and that of Dp-$\overline{\mathrm{D}p}$ system are classified by $KK^0(A, B)$. This picture is clearer when the whole space-time is a product space $A \otimes B$. For the sake of simplicity, let us explain the physical interpretation of $KK^1(A, B)$ using the non-BPS Dp-brane system. Let $C(X) = C(N) \otimes C(M)$ be a 10 dimensional space-time, where $B = C(M)$ is a $p+1$ dimensional world-volume of non-BPS Dp-branes with coordinates

[9] For notations used here and mathematical details, see [37, 8].

x^α ($\alpha = 0, 1, \ldots, p$), and $A = C(N)$ is a $9-p$ dimensional space transverse to M with coordinates y^i ($i = p+1, \ldots, 9$). Then $\mathcal{H}_B$ is the Chan-Paton bundle on M with (infinite) dimensional Hilbert spaces as fibers. The $*$-homomorphism ϕ is given by $\phi(y^i) = \Phi^i(x)$ where $\Phi^i(x)$ are the $9-p$ scalar fields on the non-BPS Dp-branes wrapped on M, and the operator $T = T_b(x)$ is the (normalized) tachyon field. They fit into odd Kasparov modules in the Fredholm picture and lead $KK^1(A, B)$. $KK^0(A, B)$ is quite analogous. In this case $\mathbb{Z}_2$ grading corresponds to Chan-Paton indices of Dp-branes and $\overline{\mathrm{D}p}$-branes. This construction generalizes the K-theory classification of D-brane charges as well as the K-homology classification of D-brane configurations given in section 4.2. In particular, it agrees with the Fredholm picture of K-theory used in [34, 47] for the $p = 9$ case.

Note that a configuration can be expanded in i-th direction and localized in α-th direction at the same time. For example, if a tachyon field is roughly given by $T \sim p_i\gamma^i + x^\alpha\gamma_\alpha$, it represents such a configuration.

In the Fredholm picture above we do not care about gauge fields, since any bundle of infinite dimensional separable Hilbert spaces is known to be trivial and the tachyon field carries the topological information instead of it. There is, however, another interesting description of Kasparov modules, the unbounded version of the Fredholm picture, which is the analog of the spectral-triple description for K-matrix theory. In this picture gauge fields can be incorporated through superconnection $Z = 1 \otimes \nabla + \sigma^1 \otimes T$ [8].

In summary, we claim that the classification based on non-BPS Dp-brane system corresponds to $KK^1(A, B)$ and that based on Dp-$\overline{\mathrm{D}p}$ system corresponds to $KK(A, B)$.

We summarize basic properties for KK-theory [37, 8]:

- $KK(A, B)$ includes both (algebraic) K-theory and (analytic) K-homology:

$$KK(\mathbb{C}, B) = K_0(B)\,, \qquad KK(A, \mathbb{C}) = K^0(A)\,. \tag{4.30}$$

- There is a bilinear associative intersection product, called Kasparov product: for any C^*-algebras A_1, A_2, B_1, B_2,

$$KK(A_1, B_1 \otimes D) \otimes_D KK(D \otimes A_2, B_2) \to KK(A_1 \otimes A_2, B_1 \otimes B_2)\,, \tag{4.31}$$

  which is essentially given by the inner tensor product of two bimodules. This especially makes $KK(A, A)$ a ring with unit element 1_A.

- Periodicity: for n even,

$$KK^n(A, B) := KK(A, B \otimes \mathbb{C}_n) \simeq KK(A, B)\,, \tag{4.32}$$

  so only $KK(A, B)$ and $KK^1(A, B)$ are independent.

- Duality: assume that there are two elements $\alpha \in KK(A \otimes B, \mathbb{C})$, $\beta \in KK(\mathbb{C}, A \otimes B)$ such that $\beta \otimes_A \alpha = 1_B \in KK(B, B)$, $\beta \otimes_B \alpha = 1_A \in KK(A, A)$. Then it follows that for any pair (D, E) of C^*-algebras the maps

$$\otimes_A \alpha : \ KK(D, A \otimes E) \to KK(D \otimes B, E)\,, \tag{4.33}$$
$$\otimes_B \alpha : \ KK(D, B \otimes E) \to KK(D \otimes A, E) \tag{4.34}$$

are isomorphisms (with inverse $\beta\otimes_B$ and $\beta\otimes_A$, respectively). Such a pair A and B are called K-dual of each other. In particular, for $D = E = \mathbb{C}$ it gives Poincaré duality between K-theory and K-homology:

$$K_*(A) \simeq K^*(B)\,, \qquad K_*(B) \simeq K^*(A)\,. \tag{4.35}$$

A simple example of this is given for $A = B = C(M)$, where M is a compact Spin^c manifold. In this case, $\alpha \in KK^n(C(M)\otimes C(M), \mathbb{C})$ ($n = \mathrm{dim}M$ mod 2) is known as the Dirac K-cycle $[M]$ and gives isomorphism (4.33).

Since non-BPS Dp-branes or Dp-$\overline{\mathrm{D}p}$ system considered above are also (unstable) configurations constructed in the K-matrix theory, the classification of D-branes built from non-BPS Dp-branes or Dp-$\overline{\mathrm{D}p}$ system should be related to that built from the K-matrix theory. This is shown by the isomorphism (4.33). For example, take the Dirac K-cycle $\alpha \in KK^n(C(M)\otimes C(M), \mathbb{C})$ with $n = \dim M$ (mod 2) and set $D = C(N)$, $E = \mathbb{C}$. Then, it gives the following isomorphism

$$KK^{i+n}(C(N), C(M)) \simeq KK^i(C(N\times M), \mathbb{C})\,, \tag{4.36}$$

where $i+n$ and i are understood mod 2. This isomorphism (4.36) generalizes the isomorphism between K-theory and K-homology (4.11). For the case $i = 0$, the right hand side of (4.36) is the group which classifies the D-branes in IIB K-matrix theory. On the other hand, the left hand side is the group which classifies the D-branes made from non-BPS Dp-branes (for n = odd) or Dp-$\overline{\mathrm{D}p}$ system (for n = even) wrapped on M. Analogous relations for type-IIA string theory holds. This means that various descriptions give the same result as expected.

The KK-theory unifies various constructions of D-branes via tachyon condensation. It would be useful to analyze some duality transformations, such as T-duality. We will come back to this issue elsewhere.

5. Boundary states and spectral triples

Given a configuration $(\mathcal{H}, \{\Phi^\mu\}, T)$ in the K-matrix theory, we can construct, at least formally, a (off-shell) boundary state that corresponds to it. There are many applications for this approach. The boundary state can be used to evaluate the BSFT action and calculate the tension and RR-charge of the D-brane represented by the spectral triple. Moreover, we can explicitly see how the spectral triples represent higher dimensional D-branes in an analogous way that is given in [9, 48] for bosonic string theory. This section is devoted to explain these results and give another viewpoint for our proposal that D-branes are represented by spectral triples. To be specific, we will examine the boundary states for type-IIA non-BPS D-instantons, though the generalization to the other cases is straightforward.

5.1 Boundary states

In this section, we review in some detail the boundary state approach for D-branes.

The boundary state (See [49] for a review.) of the non-BPS D-instanton in type-IIA string theory is given by

$$\left|\widehat{D}(-1)\right\rangle = \frac{1}{\sqrt{2}}\left(|D;+\rangle_{\rm NS} - |D;-\rangle_{\rm NS}\right). \tag{5.1}$$

Here $|D;\pm\rangle_{\rm NS}$ are boundary states which satisfy Dirichlet boundary condition for all directions and NS-fermions with $\pm$ spin structure. They can be expressed as[10]

$$|D;\pm\rangle_{\rm NS} = |x=0\rangle\,|\theta=0;\pm\rangle_{\rm NS}\ , \tag{5.2}$$

using coherent states

$$|x\rangle = \exp\left(\sum_{m=1}^{\infty}\left(-\frac{1}{2}x_{-m}x_m - a_m^\dagger\widetilde{a}_m^\dagger + a_m^\dagger x_m + x_{-m}\widetilde{a}_m^\dagger\right)\right)|x_0\rangle\ , \tag{5.3}$$

$$|\theta;\pm\rangle_{\rm NS} = \exp\left(\sum_{r>0}\left(-\frac{1}{2}\theta_{-r}\theta_r \pm i\psi_r^\dagger\widetilde{\psi}_r^\dagger + \psi_r^\dagger\theta_r \mp i\theta_{-r}\widetilde{\psi}_r^\dagger\right)\right)|0\rangle_{\rm NS}\ , \tag{5.4}$$

where

$$a_m^\mu = \frac{i\alpha_m^\mu}{\sqrt{m}}\,, \qquad a_{-m}^\mu = a_m^{\mu\dagger} = -\frac{i\alpha_{-m}^\mu}{\sqrt{m}}\,, \qquad (m>0)\,. \tag{5.5}$$

These are eigen states of operators $X^\mu(\sigma)$ and $\Theta_\pm^\mu(\sigma)$ defined as

$$X^\mu(\sigma) = \widehat{x}_0^\mu + \sum_{m\neq 0}\frac{1}{\sqrt{|m|}}\left(a_m^\mu e^{-im\sigma} + \widetilde{a}_m^\mu e^{im\sigma}\right), \tag{5.6}$$

$$\Theta_\pm^\mu(\sigma) = \psi^\mu(\sigma) \pm i\widetilde{\psi}^\mu(\sigma) = \sum_r\left(\psi_r^\mu e^{-ir\sigma} \pm i\widetilde{\psi}_r^\mu e^{ir\sigma}\right), \tag{5.7}$$

and satisfy

$$X^\mu(\sigma)\,|x\rangle = x^\mu(\sigma)\,|x\rangle\ , \tag{5.8}$$

$$\Theta_\pm^\mu(\sigma)\,|\theta;\pm\rangle = \theta^\mu(\sigma)\,|\theta;\pm\rangle\ , \tag{5.9}$$

$$\int [dx]\,|x\rangle\,\langle x| = 1\,, \tag{5.10}$$

$$\int [d\theta]\,|\theta;\pm\rangle\,\langle\theta;\pm| = 1\,, \tag{5.11}$$

where

$$x^\mu(\sigma) = x_0^\mu + \sum_{m\neq 0}\frac{1}{\sqrt{|m|}}\,x_m^\mu e^{-im\sigma}\,, \qquad \theta^\mu(\sigma) = \sum_r \theta_r^\mu e^{-ir\sigma}\,. \tag{5.12}$$

[10] We will omit the ghost part of the boundary states, which do not play an important role in our analysis.

The boundary state for the Neumann boundary condition is obtained by integrating these coherent states as

$$|N;\pm\rangle_{\rm NS} = \int [dx][d\theta]\,|x\rangle\,|\theta;\pm\rangle_{\rm NS} \tag{5.13}$$

$$= e^{+\sum_{m=1}^{\infty} a_m^{\dagger}\widetilde{a}_m^{\dagger} \mp \sum_{r>0} i\psi_r^{\dagger}\widetilde{\psi}_r^{\dagger}}\,|0\rangle_{\rm NS}\,. \tag{5.14}$$

Here the ground state is the zero momentum state $\int \frac{dx_0}{\sqrt{2\pi}}\,|x_0\rangle = |0\rangle$.

In fact, one can easily check that this state satisfies the Neumann boundary condition,

$$0 = \int [dx][d\theta]\frac{\delta}{i\delta x^{\mu}(\sigma)}\,|x\rangle\,|\theta;\pm\rangle \tag{5.15}$$

$$= \int [dx][d\theta]\frac{\delta}{i\delta x^{\mu}(\sigma)} e^{i\int d\sigma' P_{\nu}(\sigma')\,x^{\nu}(\sigma')}\,|x=0\rangle\,|\theta;\pm\rangle \tag{5.16}$$

$$= P_{\mu}(\sigma)\,|N;\pm\rangle\,, \tag{5.17}$$

where $P_{\mu}(\sigma)$ is the momentum operator conjugate to X^{μ}.

$$P_{\mu}(\sigma) = \frac{1}{2}\sum_{m=-\infty}^{\infty}\left(\alpha_{m\mu}\,e^{-im\sigma} + \widetilde{\alpha}_{m\mu}\,e^{im\sigma}\right). \tag{5.18}$$

Therefore the boundary states for a Dp-brane stretched along x^{α} ($\alpha = 0,1,\ldots,p$) axes and located at $x^i = 0$ ($i = p+1,\ldots,9$) are linear combinations of the following states.

$$|Bp;\pm\rangle = \int [dx^{\alpha}][d\theta^{\alpha}]\,\left|x^{\alpha}, x^i = 0\right\rangle\,\left|\theta^{\alpha}, \theta^i = 0;\pm\right\rangle\,. \tag{5.19}$$

We can similarly construct the boundary states in RR-sector. In this case, however, we should be careful about the fermion zero mode. The coherent state for the RR-fermion operator $\Theta_{\pm}(\sigma)$ is

$$|\theta;\pm\rangle_{\rm RR} = \exp\left(\sum_{n=1}^{\infty}\left(-\frac{1}{2}\theta_{-n}\theta_n \pm i\psi_n^{\dagger}\widetilde{\psi}_n^{\dagger} + \psi_n^{\dagger}\theta_n \mp i\theta_{-n}\widetilde{\psi}_n^{\dagger}\right)\right) e^{i\frac{1}{2}(\psi_0 \mp i\widetilde{\psi}_0)\theta_0}\,|D;\pm\rangle_{\rm RR}^{(0)}\,, \tag{5.20}$$

where $|D;\pm\rangle_{\rm RR}^{(0)}$ is the ground state defined as follows.

Let us define the zero mode operators

$$\psi_{\pm}^{\mu} = \frac{1}{\sqrt{2}}(\psi_0^{\mu} \pm i\widetilde{\psi}_0^{\mu})\,, \tag{5.21}$$

which satisfy the following anti-commutation relations

$$\{\psi_{+}^{\mu}, \psi_{-}^{\nu}\} = \delta^{\mu\nu}\,, \qquad \{\psi_{+}^{\mu}, \psi_{+}^{\nu}\} = \{\psi_{-}^{\mu}, \psi_{-}^{\nu}\} = 0\,. \tag{5.22}$$

These are nothing but the anti-commutation relations of SO(10, 10) gamma matrices which we encountered in (2.11). So we can regard $\psi_{\pm}$ as SO(10, 10) gamma matrices. $|D;\pm\rangle_{\rm RR}^{(0)}$ is one of the states which belong to the irreducible representation of the gamma matrices, and satisfies the the Dirichlet boundary condition

$$\psi_{\pm}^{\mu}\,|D;\pm\rangle_{\rm RR}^{(0)} = 0\,. \tag{5.23}$$

Similarly, the Neumann boundary condition implies

$$\psi_{\mp}^{\mu}\,|N;\pm\rangle_{\rm RR}^{(0)}=0\,, \tag{5.24}$$

and hence the ground state with a mixed boundary condition as (5.19) should satisfy

$$\psi_{\mp}^{\alpha}\,|Bp;\pm\rangle_{\rm RR}^{(0)}=0 \qquad (\alpha=0,1,\dots,p)\,, \tag{5.25}$$
$$\psi_{\pm}^{i}\,|Bp;\pm\rangle_{\rm RR}^{(0)}=0 \qquad (i=p+1,\dots,9)\,. \tag{5.26}$$

They are constructed by acting ψ_{-}^{μ} on $|D;+\rangle_{\rm RR}^{(0)}$ as

$$|Bp;+\rangle_{\rm RR}^{(0)}=\prod_{\alpha=0}^{p}\psi_{-}^{\alpha}\,|D;+\rangle_{\rm RR}^{(0)}\,, \tag{5.27}$$
$$|Bp;-\rangle_{\rm RR}^{(0)}=\prod_{\alpha=0}^{p}\psi_{+}^{\alpha}\prod_{i=p+1}^{9}\psi_{-}^{i}\,|Bp;+\rangle_{\rm RR}^{(0)}\,, \tag{5.28}$$

up to phase factor. We fix the phase difference among $|Bp;\pm\rangle_{\rm RR}^{(0)}$ by these relations.

Then one can easily show that they satisfy

$$(-1)^{F}\,|Bp;\pm\rangle_{\rm NS}=-\,|Bp;\mp\rangle_{\rm NS}\,, \tag{5.29}$$
$$(-1)^{\widetilde{F}}\,|Bp;\pm\rangle_{\rm NS}=-\,|Bp;\mp\rangle_{\rm NS}\,, \tag{5.30}$$
$$(-1)^{F}\,|Bp;\pm\rangle_{\rm RR}=|Bp;\mp\rangle_{\rm RR}\,, \tag{5.31}$$
$$(-1)^{\widetilde{F}}\,|Bp;\pm\rangle_{\rm RR}=(-1)^{p+1}\,|Bp;\mp\rangle_{\rm RR}\,, \tag{5.32}$$

where F ($\widetilde{F}$) is the world-sheet left (right) moving fermion number operator. Note that $(-1)^F$ and $(-1)^{\widetilde{F}}$ act as

$$(-1)^{F}=\prod_{\mu=0}^{p}(\psi_{+}^{\mu}+\psi_{-}^{\mu})\,, \tag{5.33}$$
$$(-1)^{\widetilde{F}}=\prod_{\mu=0}^{p}(\psi_{+}^{\mu}-\psi_{-}^{\mu}) \tag{5.34}$$

on the RR ground states.

The linear combinations that survive after GSO projection are the boundary states of non-BPS Dp-branes with odd (even) p

$$\left|\widehat{D}p\right\rangle=\frac{1}{\sqrt{2}}\left(|Bp;+\rangle_{\rm NS}-|Bp;-\rangle_{\rm NS}\right), \tag{5.35}$$

and BPS Dp-branes with even (odd) p

$$|Dp\rangle=\frac{1}{2}\left(|Bp;+\rangle_{\rm NS}-|Bp;-\rangle_{\rm NS}+|Bp;+\rangle_{\rm RR}+|Bp;-\rangle_{\rm RR}\right), \tag{5.36}$$

in type-IIA (IIB) string theory.

When we turn on the tachyon fields, gauge fields and so on, the boundary states $|Bp;\pm\rangle$ are modified as

$$
\begin{aligned}
|Bp;\pm\rangle_{S_b} &= \int [dx^\alpha][d\theta^\alpha]\, e^{-S_b(x,\theta)} \left|x^\alpha, x^i = 0\right\rangle \left|\theta^\alpha, \theta^i = 0; \pm\right\rangle && (5.37)\\
&= e^{-S_b(X,\Theta_\pm)}\, |Bp;\pm\rangle\,, && (5.38)
\end{aligned}
$$

where $S_b(X,\Theta)$ is the boundary interaction. The boundary interaction for non-BPS D9-branes in type-IIA string theory is given in [13, 15, 16] as

$$
e^{-S_b(X,\Theta)} = \int [d\eta^I] \exp\left\{ \oint d\sigma \left(\frac{1}{4}\dot\eta^I\eta^I + \sum_{k=0}^{2m} \frac{1}{2k!}(M_1 - M_0^2)^{I_1\cdots I_k}\eta^{I_1}\cdots\eta^{I_k} \right) \right\}, \tag{5.39}
$$

where $\dot\eta = \partial_\sigma\eta$ and

$$
\begin{aligned}
M_0 &= \begin{pmatrix} iA_\mu(X)\,\Theta^\mu & T(X) \\ T(X) & iA_\mu(X)\,\Theta^\mu \end{pmatrix}, && (5.40)\\
M_1 &= \begin{pmatrix} iA_\mu(X)\dot X^\mu + i\partial_\nu A_\mu(X)\,\Theta^\nu\Theta^\mu & \partial_\mu T(X)\,\Theta^\mu \\ \partial_\mu T(X)\,\Theta^\mu & iA_\mu(X)\dot X^\mu + i\partial_\nu A_\mu(X)\,\Theta^\nu\Theta^\mu \end{pmatrix}, && (5.41)\\
M_1 - M_0^2 &= \begin{pmatrix} iA_\mu\dot X^\mu - T^2 + \frac{i}{2}F_{\nu\mu}\,\Theta^\nu\Theta^\mu & D_\mu T\,\Theta^\mu \\ D_\mu T\,\Theta^\mu & iA_\mu\dot X^\mu - T^2 + \frac{i}{2}F_{\nu\mu}\,\Theta^\nu\Theta^\mu \end{pmatrix}. && (5.42)
\end{aligned}
$$

Here we assume that M_0 and M_1 are $2^m \times 2^m$ matrices, and $M^{I_1\cdots I_k}$ stand for coefficients with respect to the following expansion of a $2^m \times 2^m$ matrix M

$$
M = \sum_{k=0}^{2m} \frac{1}{2k!} M^{I_1\cdots I_k}\gamma^{I_1\cdots I_k}\,, \tag{5.43}
$$

where $\gamma^{I_1\cdots I_k}$ are the skew-symmetric products of SO$(2m)$ gamma matrices γ^I.

The boundary interaction for non-BPS D-instantons can be obtained by replacing

$$
\begin{aligned}
A_\mu &\to \Phi_\mu\,, && (5.44)\\
F_{\nu\mu} &\to i[\Phi_\nu, \Phi_\mu]\,, && (5.45)\\
D_\mu T &\to i[\Phi_\mu, T]\,, && (5.46)\\
\dot X^\mu &\to 2P^\mu\,, && (5.47)\\
\Theta_\pm^\mu &\to \Theta_\mp^\mu =: -2i\Pi_\pm^\mu\,, && (5.48)
\end{aligned}
$$

where P^μ and $\Pi_\pm^\mu$ are canonical momentum conjugate to X^μ and $\Theta_\pm^\mu$ respectively. Thus we obtain the boundary interaction (5.39) with

$$
M_1 - M_0^2 = \begin{pmatrix} i\Phi_\mu P^\mu - T^2 - \frac{1}{2}[\Phi_\nu, \Phi_\mu]\,\Pi^\nu\Pi^\mu & [\Phi_\mu, T]\,\Pi^\mu \\ [\Phi_\mu, T]\,\Pi^\mu & i\Phi_\mu P^\mu - T^2 - \frac{1}{2}[\Phi_\nu, \Phi_\mu]\,\Pi^\nu\Pi^\mu \end{pmatrix}. \tag{5.49}
$$

Here we rescaled $2\Phi_\mu \to \Phi_\mu$ to avoid factor 2 appearing everywhere.

In the case that Φ_μ and T are operators acting on an infinite dimensional Hilbert space $\mathcal{H}$, we cannot assume that M_0 and M_1 are $2^m \times 2^m$ matrices. Instead, we treat these as

2×2 matrices with operator coefficients. Namely, expanding $M_1 - M_0^2$ with respect to $I_2 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ and $\sigma^1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ as

$$M_1 - M_0^2 = \left(i\Phi_\mu P^\mu - T^2 - \frac{1}{2}[\Phi_\nu, \Phi_\mu]\,\Pi^\nu \Pi^\mu \right) I_2 + \left([\Phi_\mu T]\,\Pi^\mu \right) \sigma^1 \,, \tag{5.50}$$

we obtain the boundary interaction

$$e^{-S_b(P,\Pi)} = \int [d\eta]\, \mathrm{Tr}_{\mathcal{H}}\, \mathrm{P} \exp \left\{ \oint d\sigma \left(\frac{1}{4}\dot{\eta}\eta + i\Phi_\mu P^\mu - T^2 - \frac{1}{2}[\Phi_\nu, \Phi_\mu]\Pi^\nu \Pi^\mu + [\Phi_\mu T]\Pi^\mu \eta \right) \right\} . \tag{5.51}$$

In this equation, we should take the path ordered trace $\mathrm{Tr}_{\mathcal{H}}\,\mathrm{P}$ on the Hilbert space $\mathcal{H}$, which corresponds to integrating out the rest of η^I (except for η) in the formula (5.39).

Collecting all these together, the boundary state for non-BPS D-instantons with the boundary interaction is given as

$$\left| \widehat{D}(-1) \right\rangle_{S_b} = P \widetilde{P}_+ \, e^{-S_b(P,\Pi_+)} \, |D; +\rangle_{\mathrm{NS}} + P \widetilde{P}_- \, e^{-S_b(P,\Pi_+)} \, |D; +\rangle_{\mathrm{RR}} \,, \tag{5.52}$$

where $P = \frac{1}{2}(1 + (-1)^F)$ and $\widetilde{P}_\pm = \frac{1}{2}(1 \pm (-1)^{\widetilde{F}})$ are GSO projection operators.

It is important to note that unlike the non-BPS D-instanton state without boundary interaction (5.1), the RR-sector may not be projected out, since the boundary interaction carries the fermion zero modes. Let us check that the boundary state (5.52) becomes (5.1) when the boundary interaction is turned off, i.e. $T = \Phi_\mu = 0$. In this case, the boundary interaction for RR-sector vanishes since the zero mode of η is not saturated in the η integral. The boundary interaction for NS-sector becomes

$$e^{-S_b(P,\Pi)}/(\mathrm{Tr}_{\mathcal{H}}\ 1) = \int [d\eta]\, e^{\oint d\sigma \left(\frac{1}{4}\dot{\eta}\eta \right)} = \int \prod_{r>0} d\eta_r \, d\eta_{-r}\, e^{(\pi r \eta_r \eta_{-r})} \tag{5.53}$$

$$= \prod_{r=1/2, 1/3, \ldots} \pi r \tag{5.54}$$

$$= \sqrt{2} \,. \tag{5.55}$$

Here we used ζ-function regularization (see appendix B) in the last step.[11] Note that the factor $(\mathrm{Tr}_{\mathcal{H}}\ 1)$ represents that we have infinite number of non-BPS D-instantons when $T = 0$. Then, using (5.55), (5.29) and (5.30), (5.1) is reproduced from (5.52).

5.2 Tachyon condensation

In the last subsection, we constructed the boundary state (5.52) with the boundary interaction (5.51), which corresponds to the D-brane represented by the configuration $(\mathcal{H}, \{\Phi_\mu\}, T)$. Let us demonstrate here that the boundary state corresponding to the configuration given by (2.19) and (2.20) is equivalent to the boundary state of a Dp-brane (p: even).

[11] This formula is obtained by indirect argument in [2] and used to explain the tension of non-BPS D-branes.

Inserting the configuration

$$T = u \sum_{\alpha=0}^{p} \widehat{p}_\alpha \otimes \gamma^\alpha \tag{5.56}$$

$$\Phi^\alpha = \widehat{x}^\alpha \otimes 1 \quad (\alpha = 0, \dots, p), \qquad \Phi^i = 0 \quad (i = p+1, \dots, 9), \tag{5.57}$$

into (5.49), we obtain

$$M_1 - M_0^2 = \begin{pmatrix} i\widehat{x}_\alpha P^\alpha - u^2 \widehat{p}_\alpha^2 & iu\gamma_\alpha \Pi^\alpha \\ iu\gamma_\alpha \Pi^\alpha & i\widehat{x}_\alpha P^\alpha - u^2 \widehat{p}_\alpha^2 \end{pmatrix} \tag{5.58}$$

$$= (i\widehat{x}_\alpha P^\alpha - u^2 \widehat{p}_\alpha^2) I + (iu\Pi^\alpha)\Gamma_\alpha, \tag{5.59}$$

where $\Gamma_\alpha = \left(\begin{smallmatrix} & \gamma_\alpha \\ \gamma_\alpha & \end{smallmatrix}\right)$ are SO(p+2) gamma matrices. Then the boundary interaction becomes

$$e^{-S_b(P,\Pi)} = \int [d\eta^\alpha] \ \mathrm{Tr}_{\mathcal{H}} \, \mathrm{P} \exp\left\{ \oint d\sigma \left(\frac{1}{4}\dot{\eta}^\alpha \eta^\alpha + i\widehat{x}_\alpha P^\alpha - u^2 \widehat{p}_\alpha^2 + iu\Pi^\alpha \eta^\alpha \right) \right\} \tag{5.60}$$

$$= \int [d\eta^\alpha] \, e^{\oint d\sigma \left(\frac{1}{4}\dot{\eta}^\alpha \eta^\alpha + iu\Pi^\alpha \eta^\alpha \right)} \, \mathrm{Tr}_{\mathcal{H}} \, \mathrm{P} \, e^{-\oint d\sigma H(\widehat{p}, \widehat{x})}, \tag{5.61}$$

where we defined $H(\widehat{p}, \widehat{x}) = u^2 \widehat{p}_\alpha^2 - i\widehat{x}_\alpha P^\alpha$. $H(\widehat{p}, \widehat{x})$ can be thought of as a hamiltonian of a point particle with a kinetic term $u^2 \widehat{p}_\alpha^2$ and a potential term $-i\widehat{x}_\alpha P^\alpha$. We can rewrite (5.61) in terms of the path integral formulation using the standard formula

$$\mathrm{Tr}_{\mathcal{H}} \, \mathrm{P} \, e^{-\int d\sigma H(\widehat{p}, \widehat{x})} = \int [dx] e^{-\int d\sigma L(\dot{x}, x)}. \tag{5.62}$$

Thus we obtain

$$e^{-S_b(P,\Pi)} = \int [dx^\alpha][d\eta^\alpha] \ \exp\left\{ \oint d\sigma \left(\frac{1}{4}\dot{\eta}^\alpha \eta^\alpha - \frac{\dot{x}_\alpha^2}{4u^2} + ix_\alpha P^\alpha + iu\Pi^\alpha \eta^\alpha \right) \right\} \tag{5.63}$$

$$= \int [dx^\alpha][ud\theta^\alpha] \ \exp\left\{ \oint d\sigma \left(-\frac{1}{4u^2}(\dot{x}_\alpha^2 + \theta^\alpha \dot{\theta}^\alpha) + ix_\alpha P^\alpha + i\Pi^\alpha \theta^\alpha \right) \right\}, \tag{5.64}$$

where $\theta^\alpha = u\eta^\alpha$.

The boundary state of NS-sector is

$$e^{-S_b(P,\Pi_\pm)} \, |D; \pm\rangle_{\mathrm{NS}} = \tag{5.65}$$

$$= \int [dx^\alpha][ud\theta^\alpha] \exp\left\{ \oint d\sigma \left(-\frac{1}{4u^2}(\dot{x}_\alpha^2 + \theta^\alpha \dot{\theta}^\alpha) + ix_\alpha P^\alpha + i\Pi_\pm^\alpha \theta^\alpha \right) \right\} |x = 0\rangle \, |\theta = 0; \pm\rangle_{\mathrm{NS}} \tag{5.66}$$

$$= \int [dx^\alpha][ud\theta^\alpha] \exp\left\{ \oint d\sigma \left(-\frac{1}{4u^2}(\dot{x}_\alpha^2 + \theta^\alpha \dot{\theta}^\alpha) \right) \right\} \left| x^\alpha, x^i = 0 \right\rangle \left| \theta^\alpha, \theta^i = 0; \pm \right\rangle_{\mathrm{NS}} \tag{5.67}$$

$$\to \int [dx^\alpha][d\theta^\alpha] \left| x^\alpha, x^i = 0 \right\rangle \left| \theta^\alpha, \theta^i = 0; \pm \right\rangle_{\mathrm{NS}} \qquad (\text{as } u \to \infty) \tag{5.68}$$

$$= |Bp; \pm\rangle_{\mathrm{NS}} \, . \tag{5.69}$$

In (5.68), we used a ζ-function regularization trick

$$\prod_{r=1/2,3/2,\dots} u^2 = 1, \tag{5.70}$$

and took a naive limit. Let us justify (5.68) using oscillator expansion and ζ-function regularization. First, we consider the bosonic part. We adopt the $p = 0$ case for simplicity.

$$\begin{aligned}
&\int [dx] \exp\left(-\oint d\sigma \frac{\dot{x}^2}{4u^2}\right)|x\rangle = \\
&= \int dx_0 \prod_{n=1}^{\infty}\left(\int dx_{-n} dx_n \, e^{-\frac{n}{2u^2}x_{-n}x_n - \frac{1}{2}x_{-n}x_n - a_n^\dagger \tilde{a}_n^\dagger + a_n^\dagger x_n + x_{-n}\tilde{a}_n^\dagger}\right)|x_0\rangle \\
&= \frac{u\Gamma(u^2)}{\sqrt{2\pi}} e^{-\sum_{n=1}^{\infty}\left(1-\frac{2}{1+n/u^2}\right)a_n^\dagger \tilde{a}_n^\dagger}|0\rangle \, . \qquad (5.71)
\end{aligned}$$

We used the ζ-function regularization to obtain (5.71) (see appendix B).

Similarly the fermionic part can also be calculated.

$$\begin{aligned}
&\int [ud\theta] \exp\left(-\oint d\sigma \frac{\theta\dot{\theta}}{4u^2}\right)|\theta; \pm\rangle_{\mathrm{NS}} = \\
&= \prod_{r>0}^{\infty}\left(\int u^2 d\theta_{-r} d\theta_r \, e^{-\frac{r}{2u^2}\theta_{-r}\theta_r - \frac{1}{2}\theta_{-r}\theta_r \pm i\psi_r^\dagger \tilde{\psi}_r^\dagger + \psi_r^\dagger \theta_r \mp i\theta_{-r}\tilde{\psi}_r^\dagger}\right)|0\rangle_{\mathrm{NS}} \\
&= \frac{\sqrt{2\pi}}{\Gamma\left(u^2+1/2\right)} e^{\pm\sum_{r>0}^{\infty}\left(1-\frac{2}{1+r/u^2}\right)i\psi_r^\dagger \tilde{\psi}_r^\dagger}|0\rangle_{\mathrm{NS}} \, . \qquad (5.72)
\end{aligned}$$

Combining (5.71) and (5.72) together, we obtain

$$\begin{aligned}
e^{-S_b(P,\Pi_\pm)}|D; \pm\rangle_{\mathrm{NS}} &= \frac{u\Gamma(u^2)}{\Gamma(u^2+1/2)} e^{-\sum_{n=1}^{\infty}\left(1-\frac{2}{1+n/u^2}\right)a_n^\dagger \tilde{a}_n^\dagger \pm \sum_{r>0}^{\infty}\left(1-\frac{2}{1+r/u^2}\right)i\psi_r^\dagger \tilde{\psi}_r^\dagger}|0\rangle_{\mathrm{NS}} \\
&\to |B0; \pm\rangle_{\mathrm{NS}} \qquad (\text{as } u \to \infty) \, , \qquad (5.73)
\end{aligned}$$

which actually agrees with the previous estimation (5.69). This calculation is precisely analogous to that given in [15, 50, 51]. In fact the coefficient $\frac{u\Gamma(u^2)}{\Gamma(u^2+1/2)} = \frac{4^{u^2}\Gamma(u^2)^2}{2\sqrt{\pi}\Gamma(2u^2)}$ in (5.73) is exactly the same as the factor which plays a crucial role to obtain the exact D-brane tension in BSFT [13, 15, 16].

For the RR-sector, note that

$$[ud\theta] = ud\theta_0 \prod_{n=1}^{\infty} u^2 d\theta_{-n} d\theta_n = d\theta_0 \prod_{n=1}^{\infty} d\theta_{-n} d\theta_n \, , \qquad (5.74)$$

where we again used the ζ-function trick

$$\prod_{n=1}^{\infty} u^2 = u^{2\zeta(0)} = u^{-1} \, . \qquad (5.75)$$

Thus the same argument as in (5.68) implies that

$$e^{-S_b(P,\Pi_\pm)}|D; \pm\rangle_{\mathrm{RR}} \to |Bp; \pm\rangle_{\mathrm{RR}} \qquad (\text{as } u \to \infty) \, . \qquad (5.76)$$

More careful analysis with the ζ-function regularization as above can also be performed analogously.

$$
\begin{aligned}
&\int [ud\theta] \exp\left(-\oint d\sigma \frac{\theta\dot{\theta}}{4u^2}\right) |\theta; \pm\rangle_{\mathrm{RR}} = \\
&= \prod_{n=1}^{\infty} \left(\int u^2 d\theta_{-n} d\theta_n e^{-\frac{n}{2u^2}\theta_{-n}\theta_n - \frac{1}{2}\theta_{-n}\theta_n \pm i\psi_n^\dagger \widetilde{\psi}_n^\dagger + \psi_n^\dagger \theta_n \mp i\theta_{-n}\widetilde{\psi}_n^\dagger}\right) \int u d\theta_0 e^{i\Pi_0 \pm \theta_0} |D; \pm\rangle_{\mathrm{RR}}^{(0)} \\
&= \frac{\sqrt{2\pi}}{u\Gamma(u^2)} e^{\pm \sum_{n=1}^{\infty}\left(1-\frac{2}{1+n/u^2}\right) i\psi_n^\dagger \widetilde{\psi}_n^\dagger} |B0; \pm\rangle_{\mathrm{RR}}^{(0)} ,
\end{aligned} \tag{5.77}
$$

where we used the relation (5.27) for the ground state. The coefficient exactly cancels that in bosonic part (5.71), as expected from the quantization of the RR-charge, and we obtain

$$
\begin{aligned}
e^{-S_b(P,\Pi_\pm)} |D; \pm\rangle_{\mathrm{RR}} &= e^{-\sum_{n=1}^{\infty}\left(1-\frac{2}{1+n/u^2}\right) a_n^\dagger \widetilde{a}_n^\dagger \pm \sum_{n=1}^{\infty}\left(1-\frac{2}{1+n/u^2}\right) i\psi_n^\dagger \widetilde{\psi}_n^\dagger} |B0; \pm\rangle_{\mathrm{RR}}^{(0)} \\
&\to |B0; \pm\rangle_{\mathrm{RR}} \qquad (\text{as } u \to \infty) .
\end{aligned} \tag{5.78}
$$

Finally, (5.69) and (5.76) imply that the boundary state of non-BPS D-instantons with the boundary interactions (5.52) is exactly equal to the boundary state of BPS Dp-brane (5.36) in the limit $u \to \infty$,

$$
\left|\widehat{D}(-1)\right\rangle_{S_b} \to |Dp\rangle \qquad (\text{as } u \to \infty) , \tag{5.79}
$$

as promised.

What can we learn from this example? The boundary states we reviewed in the last subsection, such as (5.19), are constructed from the geometric information of the D-branes. Namely, we first decided how the world-volume of the D-brane is embedded in the space-time, and arranged a suitable coherent state to construct the D-brane boundary state. On the other hand, as we have seen, D-brane configurations are represented by analytic data $(\mathcal{H}, \{\Phi_\mu\}, T)$, i.e. the spectral triple, in the K-matrix theory. We can construct the D-brane boundary state corresponding to each triple $(\mathcal{H}, \{\Phi_\mu\}, T)$ as given in (5.51) and (5.52). In this subsection, we have shown, using an explicit example, that these two constructions are actually equivalent. This is the stringy realization of the isomorphism (4.15) between topological K-homology and analytic one. The key relation is (5.62), which translates the analytic data (in the operator formalism) into the geometric one (in the path integral). It is interesting to note that the Hilbert space $\mathcal{H}$, which is interpreted as the space of Chan-Paton indices of non-BPS D-instantons, is translated into the Hilbert space of the quantum mechanics of the boundary degrees of freedom defined by the path integral in the right hand side of (5.62). The analogous statement has been observed in noncommutative geometry [9, 48], but our analysis shows that this correspondence is also realized even in the commutative cases.

6. Conclusions and discussions

In this paper, we studied the matrix theory based on non-BPS D-instantons in type-IIA string theory and D-instanton–anti D-instanton system in type-IIB string theory, which we called K-matrix theory. The configurations with finite action are identified with spectral triples, and the geometry represented by the spectral triples are interpreted as the geometry on the world-volume of higher dimensional D-branes. Furthermore, we claimed that the configurations of D-branes in the K-matrix theory are classified by K-homology. We also constructed the boundary states corresponding to the configurations of the K-matrix theory, and explicitly showed that the canonical triples represent higher dimensional D-branes.

It would be interesting to investigate the relation between our proposal that D-branes are represented by the spectral triples and the description of D-branes as objects of the derived category of coherent sheaves [52, 53]. Actually, they are closely related. In fact, an element of K-homology group in algebraic geometry can be obtained by an object of the derived category of coherent sheaves [52]. Here, K-homology group in algebraic geometry, denoted $K_0'(X)$, is defined as the Grothendieck group of coherent algebraic sheaves on algebraic variety X. One can define a natural map [36]

$$\alpha : K_0'(X) \to K_0^{\rm top}(X)\,, \tag{6.1}$$

though this is *not* isomorphic in general. The reason that α fails to be isomorphic can be understood from the fact that $K_0^{\rm top}(X)$ does not respect holomorphic structure, while $K_0'(X)$ does. See [36, 52] for more detail.

There are many important issues, which we left for the future study. First of all, we have not made an argument about the consistency of the theory as a quantum theory. Since the variables in the theory are operators acting on an infinite dimensional Hilbert space, it is not clear that all the physical quantities remain finite. The action of the theory should be determined, to a certain extent, by the consistency of the theory. Similarly, we were not careful about the choice of the space-time manifold X. Since a consistent background should be a solution of the equations of motion of supergravity, there should be some restrictions for the choice. Some related arguments about the formulation of the general matrix theories in curved backgrounds can be found in [24, 25, 54].

In addition, in section 3.3 and section 4, we chose the space-time algebra $\mathcal{A}$ independent of the choice of the background, in which the K-matrix theory is supposed to be formulated. We do not know the precise relation between the closed string background and the space-time algebra we can choose. In particular, there are many D-brane configurations that are not included in the space-time represented by the fixed algebra. For example, we do have noncommutative D-brane configurations, even if the background is commutative. In the classification of stable D-brane configurations, we classified the D-brane configurations embedded in a fixed space-time manifold. There might be a possibility that the D-branes decay through going 'outside' the space-time that we fixed.

The appearance of the closed strings in the K-matrix theory is also a very interesting subject. If the theory consistently formulate the type-II string theory, there should be closed strings. Unfortunately, K-homology is not powerful enough to classify the fundamental strings and NS5-branes, and we failed to incorporate these objects into the classification. See [55] for the recent related work.

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One of the interesting features of the K-matrix theory in contrast to other matrix theories is that we can construct arbitrary numbers of the D-branes. Even the configuration with 'nothing' is also included as a configuration of the theory. This fact may have interesting applications to the formulation of M-theory. In the BFSS matrix theory [5], the number N of the D-particles are fixed to a finite or infinite value. Therefore, it can only represent M-theory with fixed momentum along the light-like or eleventh direction. On the other hand, in the K-matrix theory (based on infinite number of either non-BPS D-instantons or D0-$\overline{\text{D0}}$ pairs), there are no such restrictions, and it is quite easy to construct a configuration with arbitrary numbers of D-particles and anti D-particles. Therefore the K-matrix theory could provide a much wider framework to study M-theory.

There is another intriguing structure of the IIA K-matrix theory based on non-BPS D-instantons, which is the same structure observed in [4] for the world-volume theory of non-BPS D9-branes. Recall that the bosonic part of the K-matrix theory consists of ten scalars Φ^μ ($\mu = 0, 1, \ldots, 9$) and one tachyon T, which transforms as $10 \oplus 1$ representation under the ten dimensional Lorentz transformation. The fermions in the theory are χ_L and χ_R, which belong to left and right handed spinor representations of the Lorentz group, respectively. They coincide with the ten dimensional decomposition of the vector and spinor representations of eleven dimensional Lorentz group! Of course, it is hard to consider the tachyon as the scalar Φ^{10} which corresponds to the eleventh direction, since eleven dimensional Lorentz symmetry is broken explicitly. It would be interesting, if this fact should be a clue to a formulation of M-theory with explicit eleven dimensional Lorentz symmetry.

Although there is a possibility that the fields Φ^μ and T are sufficient to describe the whole things, the precise action of the K-matrix theory may be given by the boundary or cubic string field theory of the infinitely many non-BPS D-instantons. In such cases, infinitely many matrices or operators should be considered, and we might have to generalize the choice of the triples $(\mathcal{H}, \widehat{\mathcal{A}}, T)$, which represent the configurations.

We hope to come back to these problems in later publication.

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A. Chern-Simons term of N non-BPS D-instantons

In this appendix A, we prove the topological invariance of CS-term of N non-BPS D-instantons, namely the invariance of the CS-term under $C \to C + d\Lambda$. First we define

$$C(x) = \int dk e^{ikx} C(k) \,, \tag{A.1}$$

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and

$$J(k) = \mathrm{Tr}_2\left(\sigma^1\,\mathrm{Tr}_N\left(e^{ik\Phi+Z^2}\right)\right), \tag{A.2}$$

where

$$iZ = -i\Phi^\mu\psi_2^\mu + T\sigma^1\,, \tag{A.3}$$

$$C(x) = \sum_{n:odd} C_{\mu_1\cdots\mu_n}(x)\psi_1^{\mu_1}\cdots\psi_1^{\mu_n}\,, \tag{A.4}$$

as defined in (2.14), (2.15). Note that n is odd here.

Then we can rewrite the CS-term of N non-BPS D-instantons (2.13) as

$$S_{CS} = \mathrm{Tr}_\psi\left(\int dk\, C(k)J(k)\right). \tag{A.5}$$

In order to show that the CS-term is invariant under the gauge transformation $C \to C + d\Lambda$, it is sufficient to check that the CS-term vanishes if we take $C(k) = k_l\Lambda(k)\psi_1^l$. where $\Lambda(k) = \Lambda_{i_1\cdots i_{n-1}}\psi_1^{i_1}\cdots\psi_1^{i_{n-1}}$. This condition is indeed satisfied as

$$\begin{aligned} S_{CS} &= \mathrm{Tr}_\psi\left(\int dk\Lambda(k)k_l\psi_1^l J(k)\right) = \mathrm{Tr}_\psi\left(\int dk\Lambda(k)\frac{1}{2}\{k_l\psi_1^l, J(k)\}\right) \\ &= -\,\mathrm{Tr}_2\left(\sigma^1\,\mathrm{Tr}_\psi\left(\int dk\Lambda(k)\underline{\mathrm{Sym}}\,\mathrm{Tr}_N\left([k\Phi, Z]e^{ik\Phi+Z^2}\right)\right)\right) \\ &= i\,\mathrm{Tr}_2\left(\sigma^1\,\mathrm{Tr}_\psi\left(\int dk\Lambda(k)\,\mathrm{Tr}_N\left([e^{ik\Phi+Z^2}, Z]\right)\right)\right) = 0\,, \end{aligned} \tag{A.6}$$

where $\underline{\mathrm{Sym}}$ means symmetrization w.r.t. $[k\Phi, Z]$ and $(ik\Phi + Z^2)$. Thus we have confirmed the invariance of the CS-term under the gauge transformation of the RR-fields. Though we demonstrated the calculation for the CS-term of N non-BPS D-instanton, it is straightforward to generalize to other systems. In particular, this proof is also applicable for the CS-terms for the BPS D-branes, which has been shown in [20]. It provides a more simple and general proof for the invariance of CS-term under the gauge transformation.

Since the CS-term is a linear functional of $C(x)$, it should be written as $S_{CS} = \int_X C(x)I(x)$, where $I(x) = I^1(x) + I^2(x) + \cdots$ and I^i is a i-form determined from Φ and T through (2.13).[12] Using this form, the invariance under $C \to C + d\Lambda$ implies that $I(x)$ is closed form, i.e. it defines a cohomology class of X. Therefore we find that any configuration of Φ and T determines a cohomology class which is nothing but the RR-charge of the configuration.

B. ζ-function regularization

We summarize the zeta-function regularization formulae used in section 5. (See also [15].)

$$\prod_{r=1/2,3/2,\ldots} A = \exp\left(\log A\sum_{n=1}^{\infty}(n+1/2)^{-s}\right)\Bigg|_{s=0} = A^{\zeta(0,1/2)} = 1 \tag{B.1}$$

[12] Here we assume a smoothness of the configuration.

$$\prod_{n=1}^{\infty} A = \exp\left(\log A \sum_{n=1}^{\infty} n^{-s}\right)\Bigg|_{s=0} = A^{\zeta(0)} = A^{-1/2} \tag{B.2}$$

$$\begin{aligned}\prod_{n=1}^{\infty}(n+a)^{-1} &= \exp\left(\frac{d}{ds}\sum_{n=1}^{\infty}(n+a)^{-s}\right)\Bigg|_{s=0} \\ &= \exp\left(\frac{d}{ds}\left(\zeta(s,a)-a^{-s}\right)\right)\Bigg|_{s=0} = \frac{\Gamma(a+1)}{\sqrt{2\pi}}\end{aligned} \tag{B.3}$$

$$\prod_{r=1/2,3/2,\ldots} r = \frac{\sqrt{2\pi}}{\Gamma(1/2)} = \sqrt{2} \tag{B.4}$$

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